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Original Article

N-fuzzy biΓ-ternary semigroups

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Abstract

The notion of N-fuzzy sets (NFS) has been applied to a newly defined algebraic structure bi Γ-ternary semigroup. The notions of N-fuzzy bi Γ-ternary subsemigroups and N-fuzzy bi Γ-left (right, lateral, quasi, and bi) ideals have been defined and related properties have been investigated. The characterization of bi Γ-ternary semigroup under these ideals have been established. 2010 AMS Classifications: 08A72, 20N10, 20M12

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1. Introduction

Zadeh (1965) introduced the concept of fuzzy set. The fuzzy set theories developed by Zadeh and others have been found many applications in the domain of mathematics and elsewhere. Rosenfeld (1971) started the study of fuzzy algebraic structures and introduced fuzzy subgroup (subgroupoid) and fuzzy (left, right) ideal in his pioneering paper.

The concept of intuitionistic fuzzy set was introduced by Atanassov (1983) as a generalization of fuzzy set. Biswas (1989), introduced the concept of intuitionistic fuzzy subgroupoids. Kim and Jun (2002) applied the concept of intuitionistic fuzzy sets to the ideal theory of semigroup and defined several ideals of semigroup. Many other authors applied the concept of fuzzy set and intuitionistic fuzzy set to the algebraic structures.

A crisp set A in a universe X can be defined in the form of its characteristics function $\mu_A : X \to \{0,1\}$. So far most of the generalizations of the crisp set have been conducted on the unit interval [0,1] and they are consistent with the asymmetry observation. the generalization of the crisp set to fuzzy set and fuzzy set to intuitionistic fuzzy set relied on spreading positive information. Jun *et al.* (2009) first time used the negative meaning of information and introduced a new function which is called negative-valued function (or negative fuzzy set, briefly, *N*-fuzzy set) and constructed *N*-structures. Khan *et al.* (2009) used the idea of *N*-fuzzy sets to characterized ordered semigroups by their *N*-fuzzy ideals. Akram *et al.* (2014) worked on the *N*-fuzzy ideals of Γ-AG-groupoids.

Recently Akram *et al.* (2015) proposed a new algebraic structure called bi Γ-ternary semigroup as a generalization of Γ-semigroup and ternary semigroup. They introduced the notions of bi Γ-ternary subsemigroup, bi Γ-left (right, lateral) ideal, bi Γ-quasi ideal and bi Γ-bi-ideals for this structure and discussed the relationship between these substructures. They also defined the regular bi Γ-ternary semigroup and characterized it by these ideals.

In this paper, we have applied the concept of *N*-fuzzy set to the ideal theory of bi Γ-ternary semigroup. The notions of *N*-fuzzy bi Γ-ternary subsemigroup, *N*-fuzzy bi Γ-left (right, lateral, quasi and bi) ideals have been defined and the

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relationship between them have been investigated. The characterizations of bi Γ-ternary semigroup by these ideals have been discussed here.

2. Preliminary Concepts

Definition 2.1 (Akram *et al.*, 2015). Let $T = \{x, y, z, ...\}$ and $\Gamma = \{\alpha, \beta, \gamma, ...\}$ be two nonempty sets. Then T is called a bi Γ-ternary semigroup if it satisfies

(i) $x\alpha y\beta z \in T$

(ii) $(x\alpha y\beta z)\gamma u\delta v = x\alpha(y\beta z\gamma u)\delta v = x\alpha y\beta(z\gamma u\delta v)$, for all $x, y, z, u, v \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$.

Example 2.2 (Akram *et al.*, 2015). Let $T = \{4n+3,\ n \in N\}$ and $\Gamma = \{4n+1,\ n \in N\}$. Define the mapping $T \times \Gamma \times T \times \Gamma \times T \to T$ as $x\alpha y\beta z = x + \alpha + y + \beta + z$. Then T is a bi Γ-ternary semigroup.

Example 2.3 (Akram *et al.*, 2015). Let $T = Z^-$ and $\Gamma \subseteq Z^+$. Define $x\gamma y\delta z$, for $x, y, z \in T$ and $\gamma, \delta \in \Gamma$ as the usual multiplication of integers. Then T is a bi Γ-ternary semigroup but not a Γ-semigroup.

Example 2.4 (Akram *et al.*, 2015). Let $T = iR$, where, $i = \sqrt{-1}$ and R is the set of real numbers. If $\Gamma \subseteq R$ and $x\alpha y\beta z$ is defined as the usual multiplication of complex numbers T is a bi Γ-ternary semigroup but not a Γ-semigroup.

Definition 2.5 (Akram *et al.*, 2015) A nonempty subset A of a bi Γ-ternary semigroup T is called a bi Γ-ternary subsemigroup of T if $A\Gamma A\Gamma A \subseteq A$.

Example 2.6 (Akram *et al.*, 2015). Let $T = N = \{1, 2, 3, ...\}$ and $\Gamma = \{4n+2,\ n \in N\}$. Define $x\alpha y\beta z = x + \alpha + y + \beta + z$. Then T is a bi Γ-ternary semigroup. Let $A = \{4n, n \in N\}$ be a nonempty subset of T. Then A is a bi Γ-ternary subsemi-group of T.

Definition 2.7 (Akram *et al.*, 2015). A nonempty subset A of a bi Γ-ternary semigroup T is called a bi Γ-left ideal of T if $T\Gamma T\Gamma A \subseteq A$.

Definition 2.8 (Akram *et al.*, 2015). A nonempty subset A of a bi Γ-ternary semigroup T is called a bi Γ-right ideal of T if $A\Gamma T\Gamma T \subseteq A$.

Definition 2.9 (Akram *et al.*, 2015) A nonempty subset A of a bi Γ-ternary semigroup T is called a bi Γ-lateral ideal of T if $T\Gamma A\Gamma T \subseteq A$.

Definition 2.10 (Akram *et al.*, 2015). A nonempty subset A of a bi Γ-ternary semigroup T is called a bi Γ-ideal of T if it is a bi Γ-left, a bi Γ-right and a bi Γ-lateral ideal of T.

Definition 2.11 (Akram *et al.*, 2015) Let T be a bi Γ-ternary semigroup. A nonempty subset Q of T is called **a** bi Γ-quasi-ideal of T if

$$Q\Gamma T\Gamma T \cap T\Gamma Q\Gamma T \cap T\Gamma T\Gamma Q \subseteq Q \text{ and } Q\Gamma T\Gamma T \cap T\Gamma T\Gamma Q\Gamma T\Gamma T \cap T\Gamma T\Gamma Q \subseteq Q.$$

Definition 2.12 (Akram *et al.*, 2015). Let T be a bi Γ-ternary semigroup. A bi Γ-ternary subsemigroup B of T is called a bi Γ-bi-ideal of T if $B\Gamma T\Gamma B\Gamma T\Gamma B \subseteq B$.

Definition 2.13 (Akram *et al.*, 2015). Let $T = \{2n, n \in N\}$, $\Gamma = \{\alpha, \beta, \gamma, ...\}$ and $A = \{4n, n \in N\}$. Define, $x\alpha y\beta z = 2x + 2y + z$, for $x, y, z \in T$ and $\alpha, \beta \in \Gamma$. Then T is a bi Γ-ternary semigroup and is a bi Γ-left ideal of T but neither a bi Γ-right nor a bi-lateral ideal of T. If we define, $x\alpha y\beta z = x + 2y + 2z$ and $x\alpha y\beta z = 2x + y + 2z$ respectively, then A is a bi Γ-right and a bi Γ-lateral ideal of T.

Example 2.14 In the above example if we define, $x\alpha y\beta z = 2x + 2y + 2z$, then A is a bi Γ-left, a bi Γ-right and a bi Γ-lateral ideal of T. Hence A is a bi -ideal of T.

Proposition 2.15 (Akram *et al.*, 2015) Let *T* be a bi Γ-ternary semigroup and $\varphi \neq X \subseteq T$. Then

(i) $T\Gamma T\Gamma X$ is a bi -left ideal of

(ii) $X\Gamma T\Gamma T$ is a bi -right ideal of

(iii) $T\Gamma X\Gamma T \cup T\Gamma T\Gamma X\Gamma T\ \ \Gamma T$ is a bi Γ-lateral ideal of *T*.

3. *N*-fuzzy bi Γ-ternary subsemigroup and *N*-fuzzy bi Γ-ideals

Definition 3.1 (Khan, A negative fuzzy set (briefly, *N*-fuzzy set or NFS) in a nonempty set X is a function $\overline{\mu} : X \to [-1,0]$. Here we are using "$-$" for the negative fuzzy function.

Jun *et al.* (2009) used the term negative-valued function and *N*-function for negative fuzzy set and *N*-fuzzy set.

Definition 3.2 Let $\overline{\mu}$ be an NFS in N and $t \in [-1,0]$. Then the set $N(\overline{\mu};t) = \{x \in X \mid \overline{\mu}(x) \leq t\}$ is called an *N*-level subset of $\overline{\mu}$.

Definition 3.3 Let $\overline{\mu}_1$ and $\overline{\mu}_2$ be two NFSs in *X*. If for all $x \in X$, $\overline{\mu}_1(x) \leq \overline{\mu}_2(x)$ then $\overline{\mu}_1$ is called an *N*-fuzzy subset (NFSS) of $\overline{\mu}_2$ and is written as $\overline{\mu}_1 \leq \overline{\mu}_2$. We say that $\overline{\mu}_1 = \overline{\mu}_2$. if and only if $\overline{\mu}_1 \leq \overline{\mu}_2$ and $\overline{\mu}_2 \leq \overline{\mu}_1$.

Definition 3.4 Let $\overline{\mu}_1$ and $\overline{\mu}_2$ be two NFSs in *X*. Then their union and intersection is also an *N*-fuzzy set in *X*, defined as, for all $x \in X$,

$$(\overline{\mu}_1 \cup \overline{\mu}_2) = \{x,\ \min\{\overline{\mu}_1(x), \overline{\mu}_2(x)\}\} = (\overline{\mu}_1(x) \wedge \overline{\mu}_2(x)),$$

and

$$(\overline{\mu}_1 \cap \overline{\mu}_2) = \{x,\ \max\{\overline{\mu}_1(x), \overline{\mu}_2(x)\}\} = (\overline{\mu}_1(x) \vee \overline{\mu}_2(x)).$$

Definition 3.5 Let $X = \{a,b,c,d\}$ be a nonempty set. Define $\overline{\mu} : X \to [-1,0]$ as, $\overline{\mu}(a) = -0.7$, $\overline{\mu}(b) = -0.4$, $\overline{\mu}(c) = -0.4$, $\overline{\mu}(d) = -0.2$. Then $\overline{\mu} = \{< a,-0.7 >, < b,-0.4 >, < c,-0.3 >, < d,-0.2 >\}$, obviously, $\overline{\mu}$ is an *N*-fuzzy set in *X*.
Now, let

$$\overline{\mu}_1 = \{< a,-1 >, < b,-0.8 >, < c,-0.6 >, < d,-0.5 >\}$$

and

$$\overline{\mu}_2 = \{< a,-.7 >, < b,-0.7 >, < c,-0.5 >, < d,-0.3 >\}.$$

Then $\overline{\mu}_1$ and $\overline{\mu}_2$ are *N*-fuzzy sets in *X*. Easily we can verify that $\overline{\mu}_1 \leq \overline{\mu}_2$.
If, we take

$$\overline{\mu}_1 = \{< a,-0.8 >, < b,-0.7 >, < c,-0.5 >, < d,-0.3 >\}$$

$$\overline{\mu}_2 = \{< a,-0.7 >, < b,-0.8 >, < c,-0.6 >, < d,-0.5 >\}.$$

Then $\overline{\mu}_1$ and $\overline{\mu}_2$ are *N*-fuzzy sets in *X* and

$$\overline{\mu}_1 \cup \overline{\mu}_2 = \{< a,-0.8 >, < b,-0.8 >, < c,-0.6 >, < d,-0.5 >\}$$

$$\overline{\mu}_1 \cap \overline{\mu}_2 = \{< a,-0.7 >, < b,-0.7 >, < c,-0.5 >, < d,-0.3 >\}.$$

Obviously, $\overline{\mu}_1 \cup \overline{\mu}_2$ and $\overline{\mu}_1 \cap \overline{\mu}_2$ are *N*-fuzzy sets in X.

In what follows, let *T* denotes a bi Γ-ternary semigroup unless otherwise specified.

Definition 3.6 Let *S* be a nonempty subset of *T*. Then the *N*- characteristic function of *S* is a function $\overline{\chi}_s$ defined as, for any $x \in T$,

$$\overline{\chi}_S(x) = \begin{cases} -1, & \text{if } x \in S \\ 0, & \text{if } x \notin S. \end{cases}$$

We denotes the *N*-characteristic function of *T* by $\overline{\chi}$.

Definition 3.7 Let $\overline{\mu}_1, \overline{\mu}_2$ and $\overline{\mu}_3$ be the three NFSs in *T*. Then their product is given as, $\overline{\mu}_1 \circ_\Gamma \overline{\mu}_2 \circ_\Gamma \overline{\mu}_3$, where for any $x \in T$,

$$(\overline{\mu}_1 \circ_\Gamma \overline{\mu}_2 \circ_\Gamma \overline{\mu}_3)(x) = \begin{cases} \bigwedge_{x=a\alpha b\beta c} \{\overline{\mu}_1(a) \vee \overline{\mu}_2(b) \vee \overline{\mu}_3(c)\}, & \text{if } x = a\alpha b\beta c, \text{ for } a,b,c \in T, \alpha, \beta \in \Gamma \\ 0, & \text{otherwise.} \end{cases}$$

Definition 3.8 Let $\overline{\mu}$ be an NFS in *T*. Then $\overline{\mu}$ is called an *N*-fuzzy bi Γ-ternary subsemigroup of if

$\overline{\mu}(x\alpha y\beta z) \leq \max\{\overline{\mu}(x), \overline{\mu}(y), \overline{\mu}(z)\}$ for all $x, y, z \in T, \alpha, \beta \in \Gamma$.

Definition 3.9 Let $\overline{\mu}$ be an NFS in *T*. Then $\overline{\mu}$ is called an *N*-fuzzy bi Γ-left ideal of *T* if

$\overline{\mu}(x\alpha y\beta z) \leq \overline{\mu}(z)$, for all $x, y, z \in T, \alpha, \beta \in \Gamma$.

Definition 3.10 Let $\overline{\mu}$ be an NFS in *T*. Then $\overline{\mu}$ is called an *N*-fuzzy bi Γ-right ideal of *T* if

$\overline{\mu}(x\alpha y\beta z) \leq \overline{\mu}(x)$, for all $x, y, z \in T, \alpha, \beta \in \Gamma$.

Definition 3.11 Let $\overline{\mu}$ be an NFS in *T*. Then $\overline{\mu}$ is called an *N*-fuzzy bi Γ-lateral ideal of *T* if

$\overline{\mu}(x\alpha y\beta z) \leq \overline{\mu}(y)$, for all $x, y, z \in T, \alpha, \beta \in \Gamma$.

Definition 3.12 Let $\overline{\mu}$ be an NFS in *T*. Then $\overline{\mu}$ is called an *N*-fuzzy bi Γ-ideal of *T* if it is a bi Γ-left, a bi Γ-right and a bi Γ-lateral ideal of *T*.

Example 3.13 Let $T = Z^-$ and $\Gamma = Z^+$. Then *T* is a bi Γ-ternary semigroup but not a Γ-semigroup under the usual multiplication of integers i.e. for $x, y, z \in T$, $\alpha, \beta \in \Gamma$, $(x\alpha y\beta z) = x\alpha y\beta z$. Now define, $\overline{\mu} : T \to [-1, 0]$ as, for $x \in T$,

$$\overline{\mu}(x) = \begin{cases} -0.5, & \text{if } x \text{ is even} \\ -0.1, & \text{otherwise.} \end{cases}$$

Then $\overline{\mu}$ is an *N*-fuzzy set in *T*. By simple calculations we can verify that $\overline{\mu}$ is an *N*-fuzzy bi Γ-ternary subsemigroup of *T*.

Example 3.14 Let $T = \{a, b, c\}$ and $\Gamma = \{\alpha\}$. Then *T* is a bi Γ-ternary semigroup under the operation defined in the following table,

α	a	b	c
a	a	a	a
b	a	b	b
c	a	c	c

Define, $\overline{\mu} : T \to [-1, 0]$ such that $\overline{\mu} = \{< a, -0.3 >, < b, -0.7 >, < c, -0.5 >\}$. Then $\overline{\mu}$ is an *N*-fuzzy set in *T* which is an *N*-fuzzy bi Γ-ternary subsemigroup of *T*. Further we can verify that *A* is not an *N*-fuzzy bi Γ-left (bi Γ-right, bi Γ-lateral) ideal of *T*.

If we take $\overline{\mu} = \{< a, -0.7 >, < b, -0.5 >, < c, -0.5 >\}$, then $\overline{\mu}$ is an *N*-fuzzy bi Γ-left, a bi Γ-right and a bi Γ-lateral ideal of *T* hence a *N*-fuzzy bi Γ-ideal of *T*. Obviously it is an Γ-fuzzy bi Γ-ternary subsemigroup of *T*.

Example 3.15 Let $T = \{2n, n \in N\}$ and $\Gamma = \{\alpha, \beta, \gamma, ...\}$. Define $x\alpha y\beta z = 2x + 2y + z$, for $x, y, z \in T, \alpha, \beta \in \Gamma$, then *T* is a bi Γ-ternary semigroup. Now define, $\overline{\mu} : T \to [-1, 0]$ as

$$\overline{\mu}(x)=\begin{cases}-0.2, & \text{if } x=4n \text{ for some } n\in N\\ -0.5, & \text{otherwise.}\end{cases}$$

Then $\overline{\mu}$ is an N-fuzzy bi Γ-left ideal of T.

Now, for $2,4,6\in T,\alpha,\beta\in\Gamma$, $(2\alpha6)\beta4=2(2)+2(6)+4=20=4(5)$. Thus $\overline{\mu}(2\alpha6\beta4)=\overline{\mu}(4(5))=-0.2$ and $\overline{\mu}(2)=-0.5$ implies that $\overline{\mu}(2\alpha6\beta4)\not\leq\overline{\mu}(2)$. Hence $\overline{\mu}$ is not an N-fuzzy bi Γ-right ideal of T. Similarly, we can verify that $\overline{\mu}$ is not an N-fuzzy bi Γ-lateral ideal of T. If we define $x\alpha y\beta z=x+2y+2z$ and $x\alpha y\beta z=2x+y+2z$, then is an N-fuzzy bi Γ-right ideal of T and N-fuzzy bi Γ-lateral ideal of T, respectively. Further more if $x\alpha y\beta z=2x+2y+2z$, then $\overline{\mu}$ is an N-fuzzy bi Γ-ideal of T.

From Example 3.14 3.15, we can write the following remark.

Remark 3.16 In a bi Γ-ternary semigroup

(i) An N-fuzzy bi Γ-left (right, lateral) ideal of T is an N-fuzzy bi Γ-ternary subsemigroup of T but the converse is not true.

(ii) An N-fuzzy bi Γ-left ideal of T may not be an N-fuzzy bi Γ-right (lateral) ideal of T and vice versa.

Lemma 3.17 Let T be a bi Γ-ternary semigroup then,

(i) The intersection of any collection of N-fuzzy bi Γ-ternary subsemigroups of T is an N-fuzzy bi Γ-ternary subsemigroup of T.

(ii) The intersection of any collection of N-fuzzy bi Γ-left (right, lateral) ideals of T is an N-fuzzy bi Γ-left (right, lateral) ideal of T.

Proof. (i) Let $\{\overline{\mu}_i, i\in I\}$ be a collection of N-fuzzy bi Γ-ternary subsemigroups of T, then $\overline{\mu}_i(x\alpha y\beta z)\leq\max\{\overline{\mu}_i(x),\overline{\mu}_i(y),\overline{\mu}_i(z)\}$, for all $i\in I$, $x,y,z\in T,\alpha,\beta\in\Gamma$. Now, for all for $i\in I$, $x,y,z\in T,\alpha,\beta\in\Gamma$.

$$(\bigcap_{i\in I}\overline{\mu}_i)(xay\beta z)=\bigcap_{i\in I}\overline{\mu}_i((xay\beta z))\leq\bigcap_{i\in I}(\max\{\overline{\mu}_i(x),\overline{\mu}_i(y),\overline{\mu}_i(z)\})$$

$$=\max\{\bigcap_{i\in I}\overline{\mu}_i(x),\bigcap_{i\in I}\overline{\mu}_i(y),\bigcap_{i\in I}\overline{\mu}_i(z)\}=\max\{(\bigcap_{i\in I}\overline{\mu}_i)(x),(\bigcap_{i\in I}\overline{\mu}_i)(y),(\bigcap_{i\in I}\overline{\mu}_i)(z)\}.$$

Hence $\bigcap_{i\in I}\overline{\mu}_i$ is an N-fuzzy bi Γ-ternary subsemigroup of T.

(ii) Proof is similar as (i).

Proposition 3.18 Let $\overline{\mu}$ be an NFS in T then

(i) $\overline{\mu}$ is an N-fuzzy bi Γ-ternary subsemigroup of T if and only if, $\overline{\mu}\circ_\Gamma\overline{\mu}\circ_\Gamma\overline{\mu}\geq\overline{\mu}$.

(ii) $\overline{\mu}$ is an N-fuzzy bi Γ-left ideal of T if and only if, $\overline{\chi}\circ_\Gamma\overline{\chi}\circ_\Gamma\overline{\mu}\geq\overline{\mu}$.

(iii) $\overline{\mu}$ is an N-fuzzy bi Γ-lateral ideal of T if and only if, $\overline{\chi}\circ_\Gamma\overline{\mu}\circ_\Gamma\overline{\chi}\geq\overline{\mu}$.

(iv) $\overline{\mu}$ is an N-fuzzy bi Γ-right ideal of T if and only if, $\overline{\mu}\circ_\Gamma\overline{\chi}\circ_\Gamma\overline{\chi}\geq\overline{\mu}$.

Proof. (i) Let $\overline{\mu}$ be an N-fuzzy bi Γ-ternary subsemigroup of T and $x\in T$.

Case1. If $x\neq a\alpha b\beta c$, for $\alpha,\beta\in\Gamma$ $a,b,c\in T$, then $(\overline{\mu}\circ_\Gamma\overline{\mu}\circ_\Gamma\overline{\mu})(x)=0\geq(\overline{\mu})(x)$.

Case2. If $x=a\alpha b\beta c$, for $\alpha,\beta\in\Gamma$ and $a,b,c\in T$, then $(\overline{\mu}\circ_\Gamma\overline{\mu}\circ_\Gamma\overline{\mu})(x)=\min_{x=a\alpha b\beta c}\{\max(\overline{\mu}(a),\overline{\mu}(b),\overline{\mu}(c))\}\geq\min_{x=a\alpha b\beta c}\overline{\mu}(a\alpha b\beta c)$

$\geq\overline{\mu}(x),\forall x\in T$.

This implies that $\overline{\mu}\circ_\Gamma\overline{\mu}\circ_\Gamma\overline{\mu}\geq\overline{\mu}$.

Conversely, we suppose that $\overline{\mu}\circ_\Gamma\overline{\mu}\circ_\Gamma\overline{\mu}\geq\overline{\mu}$. Let, $\alpha,\beta\in\Gamma$, $a,b,c\in T$ and $x=a\alpha b\beta c$ then,

$$\overline{\mu}(a\alpha b\beta c)=\overline{\mu}(x)\leq(\overline{\mu}\circ_\Gamma\overline{\mu}\circ_\Gamma\overline{\mu})(x)=\min_{x=u\delta v\theta w}\{\max(\overline{\mu}(u),\overline{\mu}(v),\overline{\mu}(w))\}\leq\max(\overline{\mu}(a),\overline{\mu}(b),\overline{\mu}(c)).$$

Hence $\overline{\mu}$ is an N-fuzzy bi Γ-ternary subsemigroup of T. Proof of Similarly, we can prove (ii), (iii) and (iv).

Lemma 3.19 Let $\overline{\mu}$ be an NFS in T. Then

(i) $\overline{\chi} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\mu}$ is an N-fuzzy bi -left ideal of T.
(ii) $\overline{\mu} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\chi}$ is an N-fuzzy bi Γ-right ideal of T.
(iii) $\overline{\chi} \circ_\Gamma \overline{\mu} \circ_\Gamma \overline{\chi}$ is an N-fuzzy bi Γ-lateral ideal of T.

Proof. (i) Let $L = \overline{\chi} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\mu}$. Then $\overline{\chi} \circ_\Gamma \overline{\chi} \circ_\Gamma L = \overline{\chi} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\mu} \geq \overline{\chi} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\mu} = L$. Hence $L = \overline{\chi} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\mu}$ is an N-fuzzy bi Γ-left ideal of T. Similarly, we can prove (ii) and (iii).

Theorem 3.20 Let $\overline{\mu}$ be an NFS in T. Then $\overline{\mu}$ is an N-fuzzy bi -ternary subsemigroup of T if and only if $N(\overline{\mu};t)$ is a bi Γ-ternary subsemigroup of T, for all $t \in [-1,0]$.

Proof. Let $\overline{\mu}$ be an N-fuzzy bi Γ-ternary subsemigroup of T. Let $x,y,z \in N(\overline{\mu};t)$, where $t \in [-1,0]$ then $\overline{\mu}(x) \leq t, \overline{\mu}(y) \leq t$ and $\overline{\mu}_A(z) \leq t$. Now for $\alpha,\beta \in \Gamma$ $\overline{\mu}(x\alpha y\beta z) \leq \max\{\overline{\mu}(x),\overline{\mu}(y),\overline{\mu}(z)\} \leq t$. This implies that $x\alpha y\beta z \in N(\overline{\mu};t)$, for all $x,y,z \in N(\overline{\mu};t)$ and $\alpha,\beta \in \Gamma$. Hence $N(\overline{\mu};t)$ is a bi Γ-ternary subsemigroup of T. Conversely, we suppose that $N(\overline{\mu};t)$ is a bi Γ-ternary subsemigroup of T, for all $t \in [-1,0]$. Let $x,y,z \in T$ such that $\overline{\mu}(x) = t_x$, $\overline{\mu}(y) = t_y$ and $\overline{\mu}(z) = t_z$ with $t_x, t_y, t_z \in [-1,0]$ then $x \in N(\overline{\mu};t_x)$, $y \in N(\overline{\mu};t_y)$ and $z \in N(\overline{\mu};t_z)$. We may assume that $t_x \leq t_y \leq t_z$ then $N(\overline{\mu};t_x) \subseteq N(\overline{\mu};t_y) \subseteq N(\overline{\mu};t_z)$, which implies that $x,y,z \in N(\overline{\mu};t_z)$. Since, $N(\overline{\mu};t_z)$ is a bi Γ-ternary subsemigroup of T implies that $x\alpha y\beta z \in N(\overline{\mu};t_z)$, for $\alpha,\beta \in \Gamma$. Then $\overline{\mu}(x\alpha y\beta z) \leq t_z = \max(t_x,t_y,t_z) = \max\left(\overline{\mu}(x),\overline{\mu}(y),\overline{\mu}(z)\right)$, for all $x,y,z \in T$ and $\alpha,\beta \in \Gamma$. Hence $\overline{\mu}$ is an N-fuzzy bi Γ-ternary subsemigroup of T.

Theorem 3.21 Let $\overline{\mu}$ be an NFS in T. Then $\overline{\mu}$ is an N-fuzzy bi Γ-left (right, lateral) ideal of T if and only if $N(\overline{\mu};t)$ is a bi Γ-left (right, lateral) ideal of T, for all $t \in [-1,0]$.

Proof. Straightforward.

Theorem 3.22 A nonempty subset S of T is a bi Γ-ternary subsemigroup of T if and only if $\overline{\chi}_S$ is an N-fuzzy bi Γ-ternary subsemigroup of T.

Proof. Let be a bi Γ-ternary subsemigroup of T then $S\Gamma S\Gamma S \subseteq S$. Let $x,y,z \in T$, $\alpha,\beta \in \Gamma$ then we have following cases.

Case 1. If $x,y,z \in S$ then $x\alpha y\beta z \in S$ and hence $\overline{\chi}_S(x) = \overline{\chi}_S(y) = \overline{\chi}_S(z) = \overline{\chi}_S(x\alpha y\beta z) = -1$ implies that $\overline{\chi}_S(x\alpha y\beta z) = \max\{\overline{\chi}_S(x),\overline{\chi}_S(y),\overline{\chi}_S(z)\}$.

Case 2. If either $x \notin S$ or $y \notin S$ or $z \notin S$ then either $\overline{\chi}_S(x) = 0$ or $\overline{\chi}_S(y) = 0$ or $\overline{\chi}_S(z) = 0$. This implies that, $\max\{\overline{\chi}_S(x),\overline{\chi}_S(y),\overline{\chi}_S(z)\} = 0$ but $\overline{\chi}_S(x\alpha y\beta z) \leq 0$ implies that $\overline{\chi}_S(x\alpha y\beta z) \leq \max\{\overline{\chi}_S(x),\overline{\chi}_S(y),\overline{\chi}_S(z)\}$, for all $x,y,z \in T$, $\alpha,\beta \in \Gamma$.

Case 3. When any two of x,y,z are not in S.

Case 4. When all x,y,z are not in S.

Above both cases give the same results as in Case2. Hence $\overline{\chi}_S$ is an N-fuzzy bi Γ-ternary subsemigroup of T.

Conversely, we suppose that $\overline{\chi}_S$ is an N-fuzzy bi Γ-ternary subsemigroup of T. Let $x,y,z \in S$ and $\alpha,\beta \in \Gamma$ then $x\alpha y\beta z \in S\Gamma S\Gamma S$. By definition of $\overline{\chi}_S$, $\overline{\chi}_S(x) = \overline{\chi}_S(y) = \overline{\chi}_S(z) = -1$ implies that $\max\{\overline{\chi}_S(x),\overline{\chi}_S(y),\overline{\chi}_S(z)\} = -1$. Since $\overline{\chi}_S$ is an N-fuzzy bi Γ-ternary subsemigroup of T, $\overline{\chi}_S(x\alpha y\beta z) \leq \max\{\overline{\chi}_S(x),\overline{\chi}_S(y),\overline{\chi}_S(z)\} = -1$ implies that $\overline{\chi}_S(x\alpha y\beta z) \leq -1$ but by definition $\overline{\chi}_S(x\alpha y\beta z) \geq -1$, which implies that $\overline{\chi}_S(x\alpha y\beta z) = -1$. This gives that $x\alpha y\beta z \in S$ implies that $S\Gamma S\Gamma S \subseteq S$. Hence S is a bi Γ-ternary subsemigroup of T.

Theorem 3.23 A nonempty subset S of T is a bi Γ-left (right, lateral) ideal of T if and only if $\overline{\chi}_S$ is an N-fuzzy bi Γ-left (right, lateral) ideal of T.

Proof. Straightforward.

Definition 3.24 Let S be a nonempty subset of T and $a, b \in [-1, 0]$ with $a \leq b$. Define a N-fuzzy set $\overline{C}_s$ in T as for all $x \in T$,

$$\overline{C}_s(x) = \begin{cases} a \text{ if } x \in S \\ b \text{ if } x \notin S. \end{cases}$$

Lemma 3.25 A nonempty subset S of T is a bi Γ-ternary subsemigroup (left ideal, right ideal, lateral ideal) of T if and only if $\overline{C}_s$ is an N-fuzzy bi Γ-ternary subsemigroup (left ideal, right ideal, lateral ideal) of T.

Proof. We prove this result for bi Γ-right ideals. Let S be a bi Γ- right ideal of T and $x, y, z \in T$. If $x \in S$ then $x\alpha y\beta z \in S$ implies that $\overline{C}_S(x) = a = \overline{C}_S(x\alpha y\beta z)$. If $x \notin S$ then $\overline{C}_S(x) = b \geq \overline{C}_S(x\alpha y\beta z)$. Hence $\overline{C}_S$ is an N-fuzzy bi Γ-right ideal of T.

Conversely, we suppose that $\overline{C}_S$ is an N-fuzzy bi Γ-right ideal of T. Let $x \in S$ then $\overline{C}_S(x) = a$. For $y, z \in T$ and $\alpha, \beta \in \Gamma, \overline{C}_S(x\alpha y\beta z) \leq \overline{C}_S(x) = a$ but $\overline{C}_S(x\alpha y\beta z) \geq a$ implies that $\overline{C}_S(x\alpha y\beta z) = a$ implies that $x\alpha y\beta z \in S \Rightarrow S\Gamma T\Gamma T \subseteq S$. Hence S is a bi Γ-right ideal of T. The result for other cases is similar.

4. N-fuzzy bi Γ-quasi ideals and *N*-fuzzy bi Γ-bi-ideals

Definition 4.1 Let $\overline{\mu}$ be an NFS in T. Then $\overline{\mu}$ is called an N-fuzzy bi Γ-quasi ideal of if for all

$$\overline{\mu}(x) \leq \max\{(\overline{\mu} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\chi})(x), (\overline{\chi} \circ_\Gamma \overline{\mu} \circ_\Gamma \overline{\chi})(x), (\overline{\chi} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\mu})(x)\},$$

$$\overline{\mu}(x) \leq \max\{(\overline{\mu} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\chi})(x), (\overline{\chi} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\mu} \circ_\Gamma \overline{\mu} \circ_\Gamma \overline{\chi})(x), (\overline{\chi} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\mu})(x)\}.$$

Alternatively,

$$\overline{\mu} \leq \overline{\mu} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\chi} \vee \overline{\chi} \circ_\Gamma \overline{\mu} \circ_\Gamma \overline{\chi} \vee \overline{\chi} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\mu}$$

$$\overline{\mu} \leq \overline{\mu} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\chi} \vee \overline{\chi} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\mu} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\chi} \vee \overline{\chi} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\mu}.$$

Proposition 4.2 Every N-fuzzy bi Γ-quasi ideal of T is an N-fuzzy bi Γ-ternary subsemigroup of T.

Proof. Let $\overline{\mu}$ be an N-fuzzy bi Γ-quasi ideal of T. Since, $\overline{\mu} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\chi} \leq \overline{\mu} \circ_\Gamma \overline{\mu} \circ_\Gamma \overline{\mu}$, $\overline{\chi} \circ_\Gamma \overline{\mu} \circ_\Gamma \overline{\chi} \leq \overline{\mu} \circ_\Gamma \overline{\mu} \circ_\Gamma \overline{\mu}$ and $\overline{\chi} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\mu} \leq \overline{\mu} \circ_\Gamma \overline{\mu} \circ_\Gamma \overline{\mu}$ implies that $\overline{\mu} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\chi} \vee \overline{\chi} \circ_\Gamma \overline{\mu} \circ_\Gamma \overline{\chi} \vee \overline{\chi} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\mu} \leq \overline{\mu} \circ_\Gamma \overline{\mu} \circ_\Gamma \overline{\mu}$. Also, since $\overline{\mu}$ is a a N-fuzzy bi Γ-quasi ideal of T, so $\overline{\mu} \leq \overline{\mu} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\chi} \vee \overline{\chi} \circ_\Gamma \overline{\mu} \circ_\Gamma \overline{\chi} \vee \overline{\chi} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\mu}$. This implies that $\overline{\mu} \leq \overline{\mu} \circ_\Gamma \overline{\mu} \circ_\Gamma \overline{\mu}$. Hence by Proposition 3.18, $\overline{\mu}$ is an N-fuzzy bi Γ-ternary subsemigroup of T.

Lemma 4.3 Let $\{\overline{\mu}_i, i \in I\}$ be a collection of N-fuzzy bi Γ-quasi ideals of T then $\bigcap_{i \in I} \overline{\mu}_i$ is also an N-fuzzy bi Γ-quasi ideal of T.

Proof. Straightforward.

Lemma 4.4 Every N-fuzzy bi Γ-left (right, lateral) ideal of T is an N-fuzzy bi Γ-quasi ideal of T.

Proof. Straightforward.

Theorem 4.5 Let $\overline{\mu}$ be an NFS in T. Then $\overline{\mu}$ is an N-fuzzy bi Γ-quasi ideal of T if and only if $N(\overline{\mu}; t)$ is a bi Γ-quasi ideal of T for all $t \in [-1, 0]$.

Proof. We suppose that $\overline{\mu}$ is an N-fuzzy bi Γ-quasi ideal of T then

$$\overline{\mu} \leq \overline{\mu} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\chi} \vee \overline{\chi} \circ_\Gamma \overline{\mu} \circ_\Gamma \overline{\chi} \vee \overline{\chi} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\mu}.$$

Let $m \in N(\overline{\mu}; t)\Gamma T\Gamma T$ then $m = n\alpha y\beta z$ for $n \in N(\overline{\mu}; t)$, $y, z \in T$ and $\alpha, \beta \in \Gamma$. Since $n \in N(\overline{\mu}; t)$ implies that $\overline{\mu}(n) \leq t$. Now,

$$(\overline{\mu} \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\chi})(m) = \min_{x=n\alpha y\beta z} \{\max(\overline{\mu}(n), \overline{\chi}(y), \overline{\chi}(z)\} \leq \max(t,-1,-1) \leq t$$

Similarly, $(\overline{\chi} \circ_{\Gamma} \overline{\mu} \circ_{\Gamma} \overline{\chi})(m) \leq t$ and $(\overline{\chi} \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\mu})(m) \leq t$ implies that

$$\max\{(\overline{\mu} \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\chi})(m), (\overline{\chi} \circ_{\Gamma} \overline{\mu} \circ_{\Gamma} \overline{\chi})(m), (\overline{\chi} \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\mu})(m)\} \leq t$$

i.e. $(\overline{\mu} \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\chi} \vee \overline{\chi} \circ_{\Gamma} \overline{\mu}_{\Gamma}\overline{\chi} \vee \overline{\chi} \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\mu})(m) \leq t,$ and by supposition,

$\overline{\mu}(m) \leq (\overline{\mu} \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\chi} \vee \overline{\chi} \circ_{\Gamma} \overline{\mu} \circ_{\Gamma} \overline{\chi} \vee \overline{\chi} \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\mu})(m)$ implies that $\overline{\mu}(m) \leq t.$

This gives that $m = n\alpha y\beta z \in N(\overline{\mu};t)$ implies that $N(\overline{\mu};t)\Gamma T\Gamma T \subseteq N(\overline{\mu};t).$

So, $N(\overline{\mu};t)\Gamma T\Gamma T \cap T\Gamma N(\overline{\mu};t)\Gamma T \cap T\Gamma T\Gamma N(\overline{\mu};t) \subseteq N(\overline{\mu};t)$

and $N(\overline{\mu};t)\Gamma T\Gamma T \cap T\Gamma T\Gamma N(\overline{\mu};t)\Gamma T\Gamma T \cap T\Gamma T\Gamma N(\overline{\mu};t) \subseteq N(\overline{\mu};t).$

Hence $N(\overline{\mu};t)$ is a bi Γ-quasi ideal of *T.*

Conversely, we suppose that $N(\overline{\mu};t)$ is a bi Γ-quasi ideal of *T*, for all $t \in [-1,0]$. We have to prove that $\overline{\mu}$ is an *N*-fuzzy bi Γ-quasi ideal of *T.* On contrary, we suppose that there exist some $m \in T$ such that

$(\overline{\mu} \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\chi} \vee \overline{\chi} \circ_{\Gamma} \overline{\mu} \circ_{\Gamma} \overline{\chi} \vee \overline{\chi} \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\mu})(m) < \overline{\mu}(m),$ i.e.

$(\overline{\mu} \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\chi})(m) \vee (\overline{\chi} \circ_{\Gamma} \overline{\mu} \circ_{\Gamma} \overline{\chi})(m) \vee (\overline{\chi} \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\mu})(m) < \overline{\mu}(m).$

Now, choose a $t \in [-1,0]$ such that $(\overline{\mu} \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\chi} \vee \overline{\chi} \circ_{\Gamma} \overline{\mu} \circ_{\Gamma} \overline{\chi} \vee \overline{\chi} \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\mu})(m) \leq t < \overline{\mu}(m),$

Since, $(\overline{\mu} \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\chi})(m) \leq t$ implies that $m \in N(\overline{\mu};t)\Gamma T\Gamma T$ and $(\overline{\chi} \circ_{\Gamma} \overline{\mu} \circ_{\Gamma} \overline{\chi})(m) \leq t$ implies that $m \in T\Gamma N(\overline{\mu};t)\Gamma T$ and $(\overline{\chi} \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\mu})(m) \leq t$ implies that $m \in T\Gamma T\Gamma N(\overline{\mu};t).$ This gives that

$m \in N(\overline{\mu};t)\Gamma T\Gamma T \cap T\Gamma N(\overline{\mu};t)\Gamma T \cap T\Gamma T\Gamma N(\overline{\mu};t) \subseteq N(\overline{\mu};t) \Rightarrow m \in N(\overline{\mu};t) \Rightarrow \overline{\mu}(m) \leq t,$ which is a contradiction.

So $(\overline{\mu} \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\chi} \vee \overline{\chi} \circ_{\Gamma} \overline{\mu} \circ_{\Gamma} \overline{\chi} \vee \overline{\chi} \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\mu})(m) \geq \overline{\mu}(m), \forall m \in T.$

i.e. $(\overline{\mu} \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\chi} \vee \overline{\chi} \circ_{\Gamma} \overline{\mu} \circ_{\Gamma} \overline{\chi} \vee \overline{\chi} \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\mu}) \geq \overline{\mu}.$

Similarly, we can verify that $\overline{\mu} \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\chi} \vee \overline{\chi} \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\mu} \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\chi} \vee \overline{\chi} \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\mu} \geq \overline{\mu}.$ Hence $\overline{\mu}$ is an *N*-fuzzy bi Γ-quasi ideal of *T.*

Theorem 4.6 A nonempty subset *S* of is a bi Γ-quasi ideal of *T* if and only if $\overline{\chi}_S$ is an *N*-fuzzy bi Γ-quasi ideal of *T.*

Proof. Let *S* be a bi Γ-quasi ideal of *T.* For $m \in T$ either $m \in S$ or $m \notin S$. If $m \in S$ then $\overline{\chi}_S(m) = -1$ and

$(\overline{\chi}_S \circ_{\Gamma} \overline{\mu}_T \circ_{\Gamma} \overline{\mu}_T)(m) \geq -1$ implies that $(\overline{\chi}_S \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\chi})(m) \geq \overline{\chi}_S(m)$ implies that

$(\overline{\chi}_S \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\chi})(m) \vee \overline{\chi} \circ_{\Gamma} \overline{\chi}_S \circ_{\Gamma} \overline{\chi})(m) \vee \overline{\chi} \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\chi}_S)(m) \geq \overline{\chi}_S(m)$

i.e. $(\overline{\chi}_S \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\chi} \vee \overline{\chi} \circ_{\Gamma} \overline{\chi}_S \circ_{\Gamma} \overline{\chi} \vee \overline{\chi} \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\chi}_S)(m) \geq \overline{\chi}_S(m).$

If $m \notin S$ then either $m = a\alpha b\beta c$ or $m \neq a\alpha b\beta c,$ for $a,b,c \in T$ and $\alpha, \beta \in \Gamma$. When $m \neq a\alpha b\beta c,$ for $a,b,c \in T$ then $(\overline{\chi}_S \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\chi})(m) = 0.$ Also $\overline{\chi}_S(m) = 0$ implies that $(\overline{\chi}_S \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\chi})(m) = \overline{\chi}_S(m)$. When $m = a\alpha b\beta c,$ for $a,b,c \in T$ then maximum two of a,b,c may contained in *S* otherwise if $a,b,c \in S$ then $m = a\alpha b\beta c \in S\Gamma T\Gamma T \cap T\Gamma S\Gamma T \cap T\Gamma T\Gamma S \subseteq S$ (Since *S* is a bi Γ-quasi ideal of *T*) $\Rightarrow m \in S,$ which is a contradiction. We have following cases,

(i) $m = a\alpha b\beta c$ and $a \notin S, b \notin S, c \notin S.$ In this case $(\overline{\chi}_S \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\chi})(m) = (\overline{\chi}_S \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\chi})(a\alpha b\beta c)$
$= \min\{\max(\overline{\chi}_S(a), \overline{\chi}(b), \overline{\chi}(c)) = \max\{(0,-1,-1)\} = 0.$

This implies that $(\overline{\chi}_S \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\chi})(m) = 0.$ Similarly, $(\overline{\chi} \circ_{\Gamma} \overline{\chi}_S \circ_{\Gamma} \overline{\chi})(m) = 0$ and $(\overline{\chi} \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\chi}_S)(m) = 0.$ Since $\overline{\chi}_S(m) = 0$, which implies that $(\overline{\chi}_S \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\chi} \vee \overline{\chi} \circ_{\Gamma} \overline{\chi}_S \circ_{\Gamma} \overline{\chi} \vee \overline{\chi} \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\chi}_S)(m) = \overline{\chi}_S(m).$

(ii) $m = a\alpha b\beta c$ and exactly one of a,b,c contained in *S.* Let $a \in S, b \notin S$ and $c \notin S$ then $\overline{\chi}_S(m) = 0$ and $\overline{\chi}_S(a) = -1.$ Also $(\overline{\chi}_S \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\chi})(m) = (\overline{\chi}_S \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\chi})(a\alpha b\beta c) = \min\{\max(\overline{\chi}_S(a), \overline{\chi}(b), \overline{\chi}(c)) = \max\{(-1,-1,-1)\} = -1.$

Similarly, $(\overline{\chi} \circ_{\Gamma} \overline{\chi}_S \circ_{\Gamma} \overline{\chi})(m) = 0$ and $(\overline{\chi} \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\chi}_S)(m) = 0$ implies that

$(\overline{\chi}_S \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\chi})(m) \vee (\overline{\chi} \circ_{\Gamma} \overline{\chi}_S \circ_{\Gamma} \overline{\chi})(m) \vee (\overline{\chi} \circ_{\Gamma} \overline{\chi} \circ_{\Gamma} \overline{\chi}_S)(m) = 0 = \overline{\chi}_S(m).$

(iii) $m = a\alpha b\beta c$ and exactly two of a,b,c contained in S. Let $a \in S, b \in S$ and $c \notin S$ then $\overline{\chi}_S(m) = 0$ and $\overline{\chi}_S(a) = -1, \overline{\chi}_S(b) = -1$. In this case $(\overline{\chi}_S \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\mu}_T)(m) = -1$, $(\overline{\mu}_T \circ_\Gamma \overline{\chi}_S \circ_\Gamma \overline{\chi})(m) = -1$ and $(\overline{\chi} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\chi}_S)(m) = 0$, which implies that $(\overline{\chi}_S \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\chi})(m) \vee (\overline{\chi} \circ_\Gamma \overline{\chi}_S \circ_\Gamma \overline{\chi})(m) \vee (\overline{\chi} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\chi}_S)(m) = \overline{\chi}_S(m)$.

Hence, for all $m \in T$, $(\overline{\chi}_S \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\chi})(m) \vee (\overline{\chi} \circ_\Gamma \overline{\chi}_S \circ_\Gamma \overline{\chi})(m) \vee (\overline{\chi} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\chi}_S)(m) \geq \overline{\chi}_S(m)$.

This implies that $\overline{\chi}_S \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\chi} \vee \overline{\chi} \circ_\Gamma \overline{\chi}_S \circ_\Gamma \overline{\chi} \vee \overline{\chi} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\chi}_S \geq \overline{\chi}_S$.

Similarly, we can verify that $\overline{\chi}_S \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\chi} \vee \overline{\chi} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\chi}_S \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\chi} \vee \overline{\chi} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\chi}_S \geq \overline{\chi}_S$.

Hence, $\overline{\chi}_S$ is an N-fuzzy bi Γ-quasi ideal of T.

Conversely, we suppose that $\overline{\chi}_S$ is an N-fuzzy bi Γ-quasi ideal of T. Let $x \in S\Gamma T\Gamma T \cap T\Gamma S\Gamma T \cap T\Gamma T\Gamma S$. Then $x \in S\Gamma T\Gamma T$, $x \in T\Gamma S\Gamma T$ and $x \in T\Gamma T\Gamma S$ implies that $x = a_1\alpha_1 b_1\beta_1 c_1 = a_2\alpha_2 b_2\beta_2 c_2 = a_3\alpha_3 b_3\beta_3 c_3$, where $a_1, b_2, c_3 \in S$, $b_1, c_1, a_2, c_2, a_3, b_3 \in T$ and $\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3 \in \Gamma$. Then

$$(\overline{\chi}_S \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\chi})(x) = (\overline{\chi}_S \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\chi})(a_1\alpha_1 b_1\beta_1 c_1) = -1,\ (\overline{\chi} \circ_\Gamma \overline{\chi}_S \circ_\Gamma \overline{\chi})(x) = (\overline{\chi} \circ_\Gamma \overline{\chi}_S \circ_\Gamma \overline{\chi})(a_2\alpha_2 b_2\beta_2 c_2) = -1,$$

$$(\overline{\chi} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\chi}_S)(x) = (\overline{\chi} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\chi}_S)(a_3\alpha_3 b_3\beta_3 c_3) = -1.$$

This implies that $(\overline{\chi}_S \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\chi} \vee \overline{\chi} \circ_\Gamma \overline{\chi}_S \circ_\Gamma \overline{\chi} \vee \overline{\chi} \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\chi}_S)(x) = -1$.

Since, $\overline{\chi}_S$ is an N-fuzzy bi Γ-quasi ideal of T.

This implies that $-1 \geq \overline{\chi}_S(x)$ but $\overline{\chi}_S(x) \geq -1$ implies that $\overline{\chi}_S(x) = -1$ implies that $x \in S$ that is $S\Gamma T\Gamma T \cap T\Gamma S\Gamma T \cap T\Gamma T\Gamma S \subseteq S$.

Likewise, we can verify that $S\Gamma T\Gamma T \cap T\Gamma T\Gamma S\Gamma T\Gamma T \cap T\Gamma T\Gamma S \subseteq S$. Hence, S is a bi Γ-quasi ideal of T.

Lemma 4.7 A nonempty subset S of T is a bi Γ-quasi-ideal of T if and only if $\overline{C}_S$ is an N-fuzzy bi Γ-quasi-ideal of T.

Proof. Straightforward.

The following Examples 4.8 & 4.9, shows that the converses of Proposition 4.2 and Lemma 4.4 are not true in general.

Example 4.8 Let $T = \{a,b,c\}$ and $\Gamma = \{\alpha\}$. Then T is bi Γ-ternary semigroup along with the operation defined in the table below.

α	a	b	c
a	a	a	a
b	a	b	b
c	a	c	c

Let $A = \{b,c\}$ then A is a bi Γ-ternary subsemigroup of T but not a bi Γ-quasi ideal of T. Now, define $\overline{\mu} : T \to [-1,0]$ as $\overline{\mu}(a) = -0.4, \overline{\mu}(b) = \overline{\mu}(c) = -0.7$. Then

$$N(\overline{\mu},t) = \begin{cases} T & \text{if } t \in [-0.4,0] \\ \{b,c\} & \text{if } t \in [-0.7,-0.4) \\ \Phi & \text{if } t \in [-1,-0.7). \end{cases}$$

Obviously, $N(\overline{\mu},t)$ is a bi Γ-ternary subsemigroup of T but not a bi Γ-quasi ideal of T, for all $t \in [-1,0]$. Then by Proposition 4.2 and Theorem 4.5, $\overline{\mu}$ is an N-fuzzy bi -ternary subsemigroup of T but not an N-fuzzy bi Γ-quasi ideal of T.

Example 4.9 In above example if we take, $A = \{a,c\}$ then A is a bi Γ-quasi ideal of T. Further more A is a bi Γ-left ideal of T but neither a bi Γ-right ideal nor a bi Γ-lateral ideal of T.

Now, define $\overline{\mu} : T \to [-1,0]$ as $\overline{\mu}(a) = \overline{\mu}(c) = -0.8$, $\overline{\mu}(b) = -0.5$. Then

$$N(\overline{\mu},t)=\begin{cases} T & \text{if } t\in[-0.5,0] \\ \{a,c\} & \text{if } t\in[-0.8,-0.5) \\ \Phi & \text{if } t\in[-1,-0.8). \end{cases}$$

Obviously, $N(\overline{\mu},t)$ is a bi Γ-quasi ideal of *T*, for all $t\in[-1,0]$ but neither a bi Γ-right ideal nor a bi Γ-lateral ideal of *T* for $t\in[-0.8,-0.5)$. Hence by Theorem 4.5, $\overline{\mu}$ is an *N*-fuzzy bi Γ-quasi ideal of *T* and by Theorem 3.21, $\overline{\mu}$ is neither an *N*-fuzzy bi Γ-right ideal nor an *N*-fuzzy bi Γ-lateral ideal of *T*. Similarly, we can construct examples of *N*-fuzzy bi Γ-quasi ideal of *T*, which are not *N*-fuzzy bi Γ-left ideal of *T*.

Definition 4.10 Let $\overline{\mu}$ be an NFS in *T*. Then $\overline{\mu}$ is called an *N*-fuzzy bi Γ-bi-ideal of if *T*.

(i) $\overline{\mu}$ is an *N*-fuzzy bi Γ-ternary subsemigroup of *T*.

(ii) For all $x,y,z\in T,\ \alpha,\beta,\eta,\delta\in\Gamma,\ \overline{\mu}(x\alpha u\beta y\eta v\delta z)\le\max(\overline{\mu}_B(x),\overline{\mu}_B(y),\overline{\mu}_B(z))$.

Proposition 4.11 Let $\overline{\mu}$ be an NFS in *T*. Then $\overline{\mu}$ is an *N*-fuzzy bi Γ-bi-ideal of *T* if and only if

(i) $\overline{\mu}\circ_\Gamma\overline{\mu}\circ_\Gamma\overline{\mu}\ge\overline{\mu}$ and

(ii) $\overline{\mu}\circ_\Gamma\overline{\chi}\circ_\Gamma\overline{\mu}\circ_\Gamma\overline{\chi}\circ_\Gamma\overline{\mu}\ge\overline{\mu}$.

Proof. We suppose that $\overline{\mu}$ is an *N*-fuzzy bi Γ-bi-ideal of *T* then it is bi Γ-ternary subsemigroup of *T* and by Proposition 3.18, condition (i) holds. Now for (ii), let $m\in T$. If $m\ne x\alpha y\beta z$, for $x,y,z\in T$ and $\alpha,\beta\in\Gamma$ then $(\overline{\mu}\circ_\Gamma\overline{\chi}\circ_\Gamma\overline{\mu}\circ_\Gamma\overline{\chi}\circ_\Gamma\overline{\mu})(m)=0\ge\overline{\mu}(m)$.

If $m=x\alpha y\beta z$ and $x=u\delta v\theta w$, for $u,v,w\in T$ and $\delta,\theta\in\Gamma$ then

$$\begin{aligned}
(\overline{\mu}\circ_\Gamma\overline{\chi}\circ_\Gamma\overline{\mu}\circ_\Gamma\overline{\chi}\circ_\Gamma\overline{\mu})(m) &= \min_{m=x\alpha y\beta z}\{\max\{(\overline{\mu}\circ_\Gamma\overline{\chi}\circ_\Gamma\overline{\mu})(x),\overline{\chi}(y),\overline{\mu}(z)\}\}\\
&= \min_{m=x\alpha y\beta z}\{\max\{\min_{x=u\delta v\theta w}\{\max(\overline{\mu}(u),\overline{\chi}(v),\overline{\mu}(w))\},\overline{\chi}(y),\overline{\mu}(z)\}\}\\
&= \min_{m=x\alpha y\beta z}\{\max\{\min_{x=u\delta v\theta w}\{\max(\overline{\mu}(u),-1,\overline{\mu}(w))\},-1,\overline{\mu}(z)\}\}\\
&= \min_{m=x\alpha y\beta z}\{\min_{x=u\delta v\theta w}\{\max\{\max(\overline{\mu}(u),\overline{\mu}(w))\},\overline{\mu}(z)\}\}\\
&= \min_{m=x\alpha y\beta z}\{\min_{x=u\delta v\theta w}\{\max(\overline{\mu}(u),\overline{\mu}(w),\overline{\mu}(z)\}\}\\
&= \min_{m=u\delta v\theta w\alpha y\beta z}\{\max(\overline{\mu}(u),\overline{\mu}(w))\},\overline{\mu}(z)\}\}\\
&\ge \min_{m=u\delta v\theta w\alpha y\beta z}\overline{\mu}(u\delta v\theta w\alpha y\beta z)=\overline{\mu}(m).
\end{aligned}$$

This implies that $\overline{\mu}\circ_\Gamma\overline{\chi}\circ_\Gamma\overline{\mu}\circ_\Gamma\overline{\chi}\circ_\Gamma\overline{\mu}\ge\overline{\mu}$.

Conversely, we suppose that (i) and (ii) holds for any *N*-fuzzy subset $\overline{\mu}$ of *T*. Let $m=x\alpha u\beta y\delta v\theta z$, for $x,y,z,u,v\in T$, $\alpha,\beta,\delta,\theta\in\Gamma$. As by (ii), $\overline{\mu}\le\overline{\mu}\circ_\Gamma\overline{\chi}\circ_\Gamma\overline{\mu}\circ_\Gamma\overline{\chi}\circ_\Gamma\overline{\mu}$ implies that

$$\begin{aligned}
\overline{\mu}(x\alpha u\beta y\delta v\theta z)=\overline{\mu}(m) &\le (\overline{\mu}\circ_\Gamma\overline{\chi}\circ_\Gamma\overline{\mu}\circ_\Gamma\overline{\chi}\circ_\Gamma\overline{\mu})(m)\\
&= \min_{m=x\alpha u\beta y\delta v\theta z}\{\max\{(\overline{\mu}\circ_\Gamma\overline{\chi}\circ_\Gamma\overline{\mu})(x\alpha u\beta y),\overline{\chi}(v),\overline{\mu}(z)\}\}\\
&\le \max\{\min_{n=x\alpha u\beta y}\{\max(\overline{\mu}(x),\overline{\chi}(u),\overline{\mu}(y))\},\overline{\chi}(v),\overline{\mu}(z)\}\\
&= \min_{n=x\alpha u\beta y}\{\max\{\max(\overline{\mu}(x),-1,\overline{\mu}(y))\},-1,\overline{\mu}(z)\}\\
&\le \max(\overline{\mu}(x),\overline{\mu}(y),\overline{\mu}(z)).
\end{aligned}$$

Hence $\overline{\mu}$ is an *N*-fuzzy bi Γ-bi-ideal of *T*.

Lemma 4.12 Let $\overline{\mu}_1$ and $\overline{\mu}_2$ be two NFSs in T. Then $\overline{\mu}_1 \circ_\Gamma \overline{\chi} \circ_\Gamma \overline{\mu}_2$ is also an N-fuzzy bi Γ-bi-ideal of T.

Proof. Straightforward.

Theorem 4.13 Let $\overline{\mu}_1$, $\overline{\mu}_2$ and $\overline{\mu}_3$ be three NFSs in T. Then $\overline{\mu}_1 \circ_\Gamma \overline{\mu}_3 \circ_\Gamma \overline{\mu}_2$ is also an N-fuzzy bi Γ-bi-ideal of T if any one of $\overline{\mu}_1$, $\overline{\mu}_2$ or $\overline{\mu}_3$ is either an N-fuzzy bi Γ-left ideal or an N-fuzzy bi Γ-right ideal or an N-fuzzy bi Γ-lateral ideal of T.

Proof. Straightforward.

Lemma 4.14 Every N-fuzzy bi Γ-quasi ideal of T is an N-fuzzy bi Γ-bi-ideal of T.

Proof. Straightforward.

Lemma 4.15 Every N-fuzzy bi Γ-left (right, lateral) ideal of T is an N-fuzzy bi Γ-bi-ideal of T.

Proof. Straightforward.

Lemma 4.16 Let $\{\overline{\mu}_i, i \in I\}$ be a collection of N-fuzzy bi Γ-bi-ideals of T then $\bigcap_{i\in I} \overline{\mu}_i$ is also an N-fuzzy bi Γ-bi-ideal of T.

Proof. Straightforward.

Theorem 4.17 Let $\overline{\mu}$ be an NFS in T. Then $\overline{\mu}$ is an N-fuzzy bi Γ-bi-ideal of T if and only if $N(\overline{\mu};t)$ is a bi Γ-bi-ideal of T, for all $t \in [-1,0]$.

Proof. We suppose that $\overline{\mu}$ is an N-fuzzy bi Γ-bi-ideal of T and $m \in N(\overline{\mu};t)\Gamma T\Gamma N(\overline{\mu};t)\Gamma T\Gamma N(\overline{\mu};t)$. Then $m = n\alpha x\beta o\delta y\theta q$ for $n,o,q \in N(\overline{\mu};t)$, $x,y \in T$ and $\alpha,\beta,\delta,\theta \in \Gamma$. Since $n,o,q \in N(\overline{\mu};t)$ implies that $\overline{\mu}(n),\overline{\mu}(o),\overline{\mu}(q) \le t$. Now, since is N-fuzzy bi Γ-bi-ideal of T so $\overline{\mu}(m) = \overline{\mu}(n\alpha x\beta o\delta y\theta q) \le \max(\overline{\mu}(n),\overline{\mu}(o),\overline{\mu}(q))$ $\max(t,t,t) = t$ implies that $\overline{\mu}(m) \le t$. This implies that $m \in N(\overline{\mu};t)$ that is $N(\overline{\mu};t)\Gamma T\Gamma N(\overline{\mu};t)\Gamma T\Gamma N(\overline{\mu};t) \subseteq N(\overline{\mu};t)$. Hence $N(\overline{\mu};t)$ is a bi Γ-bi-ideal of T.

Conversely, we suppose that $N(\overline{\mu};t)$ is a bi Γ-bi-ideal of T for all $t \in [-1,0]$. Let $x,y,z \in T$ such that $\overline{\mu}(x) = t_x$, $\overline{\mu}(y) = t_y$ and $\overline{\mu}(z) = t_z$ with $t_x,t_y,t_z \in [-1,0]$. Then $x \in N(\overline{\mu};t_x), y \in N(\overline{\mu};t_y)$ and $z \in N(\overline{\mu};t_z)$. We may assume that $t_x \le t_y \le t_z$ and then $N(\overline{\mu};t_x) \subseteq N(\overline{\mu};t_y) \subseteq N(\overline{\mu};t_z)$. This implies that $x,y,z \in N(\overline{\mu};t_z)$. Since $N(\overline{\mu};t_z)$ is a bi Γ-bi-ideal of T then for $u,v \in T$, $\alpha,\beta,\eta,\theta \in \Gamma, x\alpha u\beta y\delta v\theta z \in N(\overline{\mu};t_z)$, so we have $\overline{\mu}(x\alpha u\beta y\delta v\theta z) \le t_z = \max(t_x,t_y,t_z)$ $= \max(\overline{\mu}(x),\overline{\mu}(y),\overline{\mu}(z))$.

Above holds for all $x,y,z,u,v \in T$ and $\alpha,\beta,\eta,\theta \in \Gamma$. Hence $\overline{\mu}$ is an N-fuzzy bi Γ-bi-ideal of T.

Theorem 4.18 A nonempty subset S of T is a bi Γ-bi-ideal of T if and only if $\overline{\chi}_S$ is an N-fuzzy bi Γ-bi-ideal of T.

Proof. We suppose that S is a bi Γ-bi-ideal of T then it is a bi Γ-ternary subsemigroup of T and by Theorem 3.22, $\overline{\chi}_S$ is an N-fuzzy bi Γ-ternary subsemigroup of T. Also $S\Gamma T\Gamma S\Gamma T\Gamma S \subseteq S$. Now for any $x,y,z,u,v \in T, \alpha,\beta,\eta,\theta \in \Gamma$, $x\alpha u\beta y\eta v\theta z \in T$. We have following cases,

(i) If $x,y,z \in S$ then $x\alpha u\beta y\eta v\theta z \in S\Gamma T\Gamma S\Gamma T\Gamma S \subseteq S$ implies that $\overline{\chi}_S(x) = \overline{\chi}_S(y) = \overline{\chi}_S(z) = -1 = \overline{\chi}_S(x\alpha u\beta y\eta v\theta z)$. Hence $\overline{\chi}_S(x\alpha u\beta y\eta v\theta z) = -1 = \max(\overline{\chi}_S(x),\overline{\chi}_S(y),\overline{\chi}_S(z))$.

(ii) If either $x \notin S$ or $y \notin S$ or $z \notin S$ then either $\overline{\chi}_S(x) = 0$ or $\overline{\chi}_S(y) = 0$ or $\overline{\chi}_S(z) = 0$ implies that $\max(\overline{\chi}_S(x),\overline{\chi}_S(y),\overline{\chi}_S(z)) = 0$. But $\overline{\chi}_S(x\alpha u\beta y\eta v\theta z) \le 0$. This implies that $\overline{\chi}_S(x\alpha u\beta y\eta v\theta z) \le \max(\overline{\chi}_S(x),\overline{\chi}_S(y),\overline{\chi}_S(z))$.

(iii) If any two of $x \notin A$ are not in S. It is same like (ii).

(iv) If $x \notin A$ and $y \notin A$ and $z \notin A$. It is also same like (ii).

Hence $\overline{\chi}_S$ is an N-fuzzy bi Γ-bi-ideal of T.

Conversely, we suppose that $\overline{\chi}_s$ is an *N*-fuzzy bi Γ-bi-ideal of *T*. For any $t \in S\Gamma T\Gamma S\Gamma T\Gamma S$ there exists $x, y, z \in S, u, v \in T$ and $\alpha, \beta, \eta, \theta \in \Gamma$ such that $t = x\alpha u\beta y\eta v\theta z$. Then $\overline{\chi}_s(x) = \overline{\chi}_s(y) = \overline{\chi}_s(z) = -1$ implies that $\max(\overline{\chi}_s(x), \overline{\chi}_s(y), \overline{\chi}_s(z)) = -1$. Since $\overline{\chi}_s$ is an *N*-fuzzy bi Γ-bi-ideal of *T* implies that $\overline{\chi}_s(x\alpha u\beta y\eta v\theta z) \leq \max(\overline{\chi}_s(x), \overline{\chi}_s(y), \overline{\chi}_s(z)) = -1$. But by definition $\overline{\chi}_s(x\alpha u\beta y\eta v\theta z) \geq -1$. This gives that $\overline{\chi}_s(x\alpha u\beta y\eta v\theta z) = -1$ implies that $t = x\alpha u\beta y\eta v\theta z \in S$. Which shows that $S\Gamma T\Gamma S\Gamma T\Gamma S \subseteq S$. Hence is *S* a bi Γ-bi-ideal of *T*.

Lemma 4.19 A nonempty subset *S* of *T* is a bi Γ-bi-ideal of *T* if and only if is an *N*-fuzzy bi Γ-bi-ideal of *T*.

Proof. Straightforward.

Example 4.20 Let $T = \{a, b, c\}$ and $\Gamma = \{\alpha\}$. Then *T* is bi Γ-ternary semigroup along with the operation defined in the bellow table.

α	a	b	c
a	a	a	a
b	a	b	b
c	a	c	c

Define, $\overline{\mu} : T \to [-1, 0]$ such that $\overline{\mu} = \{< a, -0.9 >, < b, -0.7 >, < c, -0.5 >\}$. Then is an *N*-fuzzy bi Γ-bi-ideal of *T*. Also, $\overline{\mu}(c\alpha c\alpha b) = \overline{\mu}(c) = -0.5 \nleq -0.7 = \overline{\mu}(b)$ and $\overline{\mu}(c\alpha b\alpha c) = \overline{\mu}(c) = -0.5 \nleq -0.7 = \overline{\mu}(b)$.
This implies that $\overline{\mu}$ is neither *N*-fuzzy bi Γ-left nor *N*-fuzzy bi Γ-lateral ideal of *T*.

Example 4.21 Let $T = Z^-$ and $\Gamma = Z^+$. Then *T* is a bi Γ-ternary semigroup but not a Γ-semigroup under the usual multiplication of integers i.e. for $x, y, z \in T$, $\alpha, \beta \in \Gamma$, $(x\alpha y\beta z) = x\alpha y\beta z$. Define, $\overline{\mu} : T \to [-1, 0]$ as, for $x \in T$,

$$\overline{\mu}(x) = \begin{cases} -0.7, & \text{if } x \text{ is even} \\ -0.1, & \text{otherwise.} \end{cases}$$

Then $\overline{\mu}$ is an *N*-fuzzy set in *T*. By simple calculations we can verify that $\overline{\mu}$ is an *N*-fuzzy bi Γ-bi-ideal of *T* as well as *N*-fuzzy bi Γ-ideal of *T*.

Example 4.22 Let

$$T = \left\{ \begin{pmatrix} a & b & c \\ 0 & 0 & d \\ 0 & 0 & e \end{pmatrix}, \ a, b, c, d, e \in Z_0^- \right\} \text{ and } \Gamma = Z^+,$$

where Z_0^- is the set of all non-positive integers. Then *T* is bi Γ-ternary semigroup under the usual and scalar multiplication of matrices. Now consider

$$B = \left\{ \begin{pmatrix} 0 & m & 0 \\ 0 & 0 & m \\ 0 & 0 & 0 \end{pmatrix}, \ m \in Z_0^- \right\}.$$

Then *B* is a bi Γ-bi-ideal of *T* but not a bi Γ-quasi ideal of *T*, as we can see below, for $s \in B, x, y, z \in T$ and $\alpha, \beta \in \Gamma$, where

$$s = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, z = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$\alpha = 1, \beta = 2,$ then $s\alpha x\beta y = z \in B\Gamma T\Gamma T, x\alpha s\beta y = z \in T\Gamma B\Gamma T$ and $x\alpha y\beta s = z \in T\Gamma T\Gamma B.$

This implies that $z \in B\Gamma T\Gamma T \cap T\Gamma B\Gamma T \cap T\Gamma T\Gamma B$ but $z \notin B$ implies that $B\Gamma T\Gamma T \cap T\Gamma B\Gamma T \cap T\Gamma T\Gamma B \not\subseteq B.$ Hence, *B* is not a bi Γ-bi-ideal of *T.* Then by Theorem 4.6, $\overline{\chi}_S$ is not an *N*-fuzzy bi Γ-quasi ideal of *T* but by Theorem 4.18, is an *N*-fuzzy bi Γ-bi-ideal of *T*. If, we define

$$\overline{C}_B(x) = \begin{cases} -0.7 \text{ if } x \in B \\ -0.2 \text{ if } x \notin B. \end{cases}$$

Then by Lemma 4.7, $\overline{C}_B(x)$ is not an *N*-fuzzy bi Γ-quasi ideal of *T* but by Lemma 4.19, $\overline{C}_B(x)$ is an *N*-fuzzy bi Γ-bi-ideal of *T.*

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