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Computing ruin probability and minimum initial capital by simulation

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Abstract

In this paper, we propose a new approximation method to obtain the ruin probability by modifying the Pollaczek-Khinchin approximation. The proposed approximation is simpler and requires fewer assumptions than other methods mentioned in the literature. The results from a simulation study show that, in some cases, the proposed method gave better ruin probability values in terms of the overall deviation from the exact values. Insurance companies are interested in calculating the initial capital by using ruin probability and so, with this in mind, we applied the proposed method to estimate the minimum initial capital that must be reserved to ensure that the ruin probability does not exceed an acceptable quantity. To illustrate the performance of our approximation, we estimated the ruin probability and the minimum initial capital with real data from an insurance company.

Keywords: risk theory, surplus process, claim process, infinite time, insurance

1. Introduction

This study is concerned with the probability of ruin in a classical compound Poisson continuous time surplus process. The surplus process at time t defined as

$$U(t) = u + ct - S(t), \tag{1}$$

where u is the initial capital, c is the rate of premium income per unit of time, and the aggregate claim process $S(t) = X_1 + X_2 + ... + X_{N(t)}$, where $N(t)$ is the number of claims at time t. The number of claims process $\{N(t); t \geq 0\}$ is assumed to be a Poisson process with intensity $\lambda > 0$. The sequence of claim sizes $\{X_i; i = 1, 2, \ldots, N(t)\}$ is assumed to be a sequence of positive independent and identically distributed (i.i.d.) random variables with distribution function F_X and a finite mean $E[X_i] = p_1$, and are independent of $N(t)$. The premium rate c is calculated using the expected value premium principle, i.e.

$$c = (1+\theta)\lambda p_1, \tag{2}$$

where $\theta > 0$ is the relative security loading. The risk of insolvency, which is when the surplus of an insurance company becomes less than zero with a given initial capital u or the probability of ruin over infinite time, is defined as

$$\psi(u) = Pr\left(U(t) < 0, \;\; for \;\; some \;\; t > 0 \middle| U(0) = u\right). \tag{3}$$

The ruin probability when the surplus process is based upon a compound Poisson aggregate claims process with the claim amounts distribution being exponential with mean $p_1 = 1/\beta$ or $Expo(\beta)$ is in the form

$$\psi(u) = \frac{1}{1+\theta}\exp\left(\frac{-\theta\beta u}{1+\theta}\right) \tag{4}$$

for all initial capital $u \geq 0$. See pp. 414-415 in Bowers *et al.* (1997) for details. For the claim amounts distributed as gamma with shape 2 and scale $1/\beta$ or $Gamma(2, \beta)$, the ruin probability in Yuanjian *et al.* (2003) is derived as follows:

$$\psi(u) = -\left[\frac{v_2(v_1+\beta)^2}{(v_1-v_2)\beta^2}e^{v_1 u} + \frac{v_1(v_2+\beta)^2}{(v_2-v_1)\beta^2}e^{v_2 u}\right], \tag{5}$$

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where $v_1 = \left(\lambda - 2c\beta + \sqrt{\lambda^2 + 4c\beta\lambda}\right)/(2c)$ and

$v_2 = \left(\lambda - 2c\beta - \sqrt{\lambda^2 + 4c\beta\lambda}\right)/(2c)$.

However, for other claim amounts distributions, ruin probabilities are not easy to obtain. Thus, approximations for ruin probability are of interest. There are many approximation methods to obtain the ruin probability, but only three are discussed here: the De Vylder approximation in De Vylder (1978), the Bowers approximation in Bowers *et al.* (1997), and the Pollaczek-Khinchin approximation in Asmussen and Binswanger (1997).

1.1 The De Vylder approximation

This approximation is based on the idea of replacing the surplus process with a surplus process of exponentially distributed claim amounts so that the first three moments coincide. Subsequently,

$$\psi_{DV}(u) = \frac{1}{1+\tilde{\theta}} \exp\left(\frac{-\tilde{\theta}\tilde{\beta}u}{1+\tilde{\theta}}\right), \tag{6}$$

where $\tilde{\theta} = (2p_1p_3\theta)/(3p_2^2)$, $\tilde{\beta} = 3p_2/p_3$ and $p_j = E\left[X_i^j\right]$, for $j = 1,2,3$. When the claim amounts distribution is exponential, $\psi_{DV}(u)$ is equal to $\psi(u)$. It should be noted here that the De Vylder approximation requires the existence of the first three moments of the claim amounts distribution.

1.2 The Bowers approximation

The well-known Lundberg upper bound of ruin probability is defined from the right hand side of the inequality below:

$$\psi(u) \le e^{-Ru} \tag{7}$$

for any initial capital $u \ge 0$, and the adjustment coefficient R is defined as the smallest positive root of

$$M_{S(t)-ct}(r) = E\left[e^{r(S(t)-ct)}\right] = e^{-rct}M_{S(t)}(r) = 1. \tag{8}$$

The ruin probability $\psi(u)$ is a non-increasing function in u, so that its lower bound, defined on p. 415 in Bowers *et al.* (1997), is

$$\psi(0) = \frac{1}{1+\theta}. \tag{9}$$

The Bowers approximation uses the fact that $(1+\theta)^{-1} \le \psi(u) \le e^{-Ru}$, giving the approximated ruin probability as

$$\psi_B(u) = \frac{1}{1+\theta}e^{-Ku}. \tag{10}$$

To obtain a reasonable constant K, the process of aggregate claims over premiums received $\{S(t)-ct\}$ is considered. Let $\{T_i; i \ge 1\}$ be the sequence of timing of claims corresponding to claim amounts $\{X_i; i \ge 1\}$. Thus, the process $\{S(t)-ct\}$ decreases with slope c and jumps at each T_i for $i = 1,2,3,\ldots$ Let M be the number of claims where the process $\{S(t)-ct\}$ becomes maximum at T_M. Let Y_1 be the value of the process $\{S(t)-ct\}$ that reaches above zero for the first time. Next, let Y_2 be the value of Y_1 excess that reaches above the value of Y_1 for the first time. Variables $Y_3, Y_4, \ldots$ are sequentially defined in the same way. Let N be the number of iterations of the process $\{S(t)-ct\}$ carried out in this sequence. It is obvious that $N \le M$. Thus, a maximal of the process $\{S(t)-ct\}$ or the maximal aggregate loss L is illustrated as

$$L = \max_{t\ge0}\{S(t)-ct\} = Y_1 + Y_2 + \ldots + Y_N. \tag{11}$$

The values Y_k, for $k = 1,2,\cdots,N$, are called new record highs and the number N is called the number of new record highs. Figure 1 shows a graph of L for $M = 5$ and $N = 3$.

Since a stationary and independent increment of process $S(t)$ is assumed, $\{Y_k\}$ is a sequence of independent and identically distributed variables with the density

$$f_Y(y) = \overline{F}_X(y)/p_1, \tag{12}$$

where $\overline{F}_X(y) = 1 - F_X(y)$ and p_1 is the expected value of the claim amounts. The number of new record highs N is geometric distributed with parameter $1-\psi(0)$, and its probability mass function is

$$\Pr(N = n) = [1-\psi(0)][\psi(0)]^n = \theta\left(\frac{1}{1+\theta}\right)^{n+1}, \tag{13}$$

where $n = 0,1,2,\ldots$ The ruin probability in infinite horizon time in (3) can be represented in the form of a distribution function of L derived as follows:

$$\psi(u) = Pr\left(\max_{t\ge0}\{S(t)-ct\} > u\right) = \Pr(L > u) = 1 - F_L(u), \tag{14}$$

where F_L is a distribution function of the maximal aggregate loss L. The constant K in (10) is chosen such that the approximated value conforms to

$$E[L] = \int_0^\infty [1-F_L(u)]\,du = \int_0^\infty \psi(u)\,du = \frac{p_2}{2\theta p_1}. \tag{15}$$

Thus,

$$\frac{p_2}{2\theta p_1} = \int_0^\infty \psi_B(u)\,du = \int_0^\infty \frac{1}{1+\theta}e^{-Ku}du.$$

So that reasonable K is

$$K = \frac{2\theta p_1}{(1+\theta)p_2}. \tag{16}$$

and the Bowers approximation in (10) becomes

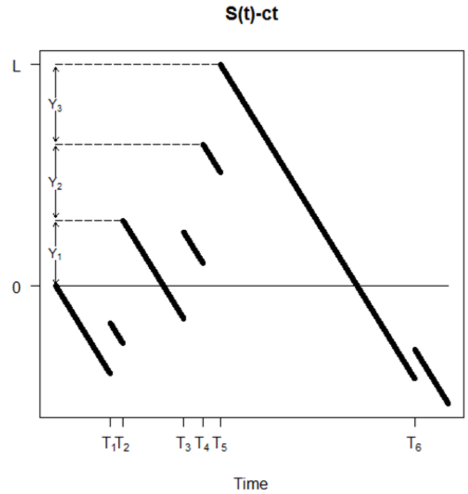

Figure 1. Maximal aggregate loss when the number of new record highs $N = 3$.

$$\psi_B(u) = \frac{1}{1+\theta} e^{\frac{-2\theta p_1}{(1+\theta)p_2}u}.$$

See pp. 418-423 in Bowers *et al.* (1997) for details. One advantage of the Bowers approximation over the De Vylder approximation is that it requires only the first two moments of the claim amounts distribution.

1.3 The Pollaczek-Khinchin approximation

This algorithm only requires the first moment of the claim amounts distribution, and the method is based on a Monte Carlo simulation using (14). To obtain $1-F_L(u)$ in (14), first define the density f_Y as in (12) and generate the number of new record highs N with a density as in (13). Next, generate a sequence of new record highs $\{Y_1, Y_2, \ldots, Y_N\}$ with f_Y. Let $L = Y_1 + Y_2 + \ldots + Y_N$ and define an indicator Z as

$$Z = \begin{cases} 0 & ; L \le u, \\ 1 & ; L > u, \end{cases} \qquad (17)$$

where $E[Z] = \psi(u)$. Repeat this process n times, so we have $Z_1, Z_2, \ldots, Z_n$ and $\overline{Z} = \sum^{n} Z_i / n$ converges to $\psi(u)$ as n is large. The algorithm for computing the approximation of the ruin probability can be presented as follows:

1. Assume F_X is known. Obtain the density f_Y from $f_Y(y) = [1 - F_X(y)] / p_1$.
2. Set the number of iterations n to be some large number. Generate $N_i, i = 1, 2, \ldots, n$ from $Geometric(q)$, where $q = \theta/(1+\theta)$, and set it to be the number of new record highs.
3. Generate Y_j^i, for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, N_i$, from the density f_Y of step 1 and obtain $L_i = Y_1^i + \ldots + Y_{N_i}^i$.
4. For each i, if $L_i > u$, then $Z_i = 1$, otherwise $Z_i = 0$.
5. Calculate $\overline{Z} = \sum_{i=1}^{n} Z_i / n$.
6. Increase the number of iterations n and repeat steps 1 to 5 until $\overline{Z}$ remains constant.

The above algorithm describes the steps for the Pollaczek-Khinchin approximation denoted by $\psi_{PK}(u) = \overline{Z}$. One difficulty of this method is at the step of simulating $\{Y_1^i, Y_2^i, \ldots, Y_{N_i}^i\}$ with density f_Y. For example, when the claim amounts distribution is $Gamma(\eta, \beta)$ with shape parameter η as an integer, the density f_Y can be derived to the density of a mixture of η gamma distributions with equal weights $1/\eta$, scale parameter β, and shape parameters $\{1, 2, \ldots, \eta\}$. However, when shape parameter η is non-integer, to simulate the amount of each new record high $\{Y_1^i, Y_2^i, \ldots, Y_{N_i}^i\}$ with f_Y is complicated. In this study, we propose a simple algorithm to approximate $\psi(u)$ based on the amount of each new record high $\{Y_1^i, Y_2^i, \ldots, Y_{N_i}^i\}$.

2. The Proposed Modified Ruin Probability Approximation

The main objective of this study is to present an algorithm to simulate the amount of each new record high $\{Y_1^i, Y_2^i, \ldots, Y_{N_i}^i\}$ for the i^{th} loop without density f_Y and apply this algorithm to real data. We represent the timing of claims in the form of $T_n = W_1 + \ldots + W_n$, $n = 1, 2, 3, \ldots$, and W_n is the time difference between consecutive claims T_n and T_{n-1}. For the Poisson claims number process with intensity $\lambda > 0$, $\{W_1, W_2, \ldots\}$ is a sequence of i.i.d. random variables with $Expo(\lambda)$.

To simulate the amount of the first new record high Y_1, we generate the time difference between consecutive claims $\{W_1, W_2, \ldots\}$ and the claim amount random variables $\{X_1, X_2, \ldots\}$. We compute the timing of claims $T_j = W_1 + \ldots + W_j$ and the value of process $\{S(t) - ct\}$, for $t = T_1, T_2, \ldots$, until the process $\{S(t) - ct\}$ reaches above the zero level for the first time.

However, it is possible that the process $\{S(t) - ct\}$, which computes forms $\{W_1, W_2, \ldots\}$ and $\{X_1, X_2, \ldots\}$, does not reach above the zero level at any time t. To mitigate this, we set a large positive integer D to be the limit on the number of simulated claims; the constant D also refers to the number of elements to truncate. Let T_D be the timing of the claim that correspond to D. When the process $\{S(t) - ct; 0 < t \le T_D\}$ reaches above the zero level at least once, then let $Y_{1,D}$ be the value of the process $\{S(t) - ct; 0 < t \le T_D\}$ that reaches above zero for the first time.

Theorem 1 If the number of truncated elements D is large, then $Y_{1,D}$ converges in distribution to the first order of new record highs Y_1.

From Theorem 1, we can approximate Y_1 by $Y_{1,D}$. The sequence of new record highs $\{Y_1, Y_2, \ldots, Y_N\}$ is an i.i.d. sequence of random variables, and so $Y_{1,D}$ converges in distribution to Y_1, for $i = 2, 3, \ldots, N$, when D is large. To simulate the amount of each new record high $\{Y_1^i, Y_2^i, \ldots, Y_{N_i}^i\}$ for the i^{th} loop, generate the time difference between consecutive claims $\{W_1, W_2, \ldots, W_D\}$ from i.i.d. $Exp(\lambda)$ and the claim amount random variables $\{X_1, X_2, \ldots, X_D\}$ with F_X. We compute the timings of the claims $\{T_1, T_2, \ldots, T_D\}$ from $T_j = W_1 + \ldots + W_j$ and let $Y_{1,D}^i$ be the value of the process $\{S(t) - ct; 0 \le t \le T_D\}$ that reaches above zero for the first time. If the process $\{S(t) - ct\}$ does not reach above zero for all $0 < t \le T_D$, then repeatedly generate the timings of the claims and the claim amount random variables until $Y_{1,D}^i$ occurs. We approximate Y_1^i by $Y_{1,D}^i$ and repeat this process until the values $\{Y_{2,D}^i, Y_{3,D}^i, \ldots, Y_{N_i,D}^i\}$ that approximate $\{Y_2^i, Y_3^i, \ldots, Y_{N_i}^i\}$ are obtained. In a real situation where the claim amounts distribution F_X is unknown, we would approximate F_X based on real claim amounts data. The proposed algorithm to obtain an approximation of the ruin probability is as follows:

1. Approximate F_X based on real data.
2. Set the number of iterations n and number of truncated elements D to be some large number. Generate $N_i, i = 1, 2, \ldots, n$ from $Geometric(q)$, where $q = \theta / (1+\theta)$, and set them to be the number of new record highs.
3. Generate sequence $\{W_1, W_2, \cdots, W_D\}$ from i.i.d. $Exp(\lambda)$ and $\{X_1, X_2, \ldots, X_D\}$ with F_X. Let $T_j = W_1 + \ldots + W_j$ and $S_j = X_1 + \ldots + X_j$ be the timings of the claims and the values of the claims process, respectively. Compute the value of the process $\{S(t) - ct; 0 \le t \le T_D\}$ by $V_j = S_j - cT_j$, for $j = 1, 2, \ldots, D$.
4. If $V_j > 0$ for some $j = 1, 2, \ldots, D$, then let $Y_{1,D}^i$ be the first V_j above zero, else repeat step 3.
5. Obtain the amount of $\{Y_{2,D}^i, Y_{3,D}^i, \ldots, Y_{N_i,D}^i\}$ by repeating steps 2 to 4 and let $L_{i,D} = Y_{1,D}^i + \ldots + Y_{N_i,D}^i$.
6. For each i, if $L_{i,D} > u$, then $Z_i = 1$, otherwise $Z_i = 0$.
7. Calculat e $\bar{Z} = \sum_{i=1}^{n} Z_i / n$.
8. Increase n and D, and repeat steps 1 to 5 until $\bar{Z}$ remains constant.

We use the proposed approximation $\psi_M(u) = \bar{Z}$ to approximate $E[Z_i] = Pr(L_{i,D} > u)$. From Theorem 1 and the fact that $\{Y_1^i, Y_2^i, \ldots, Y_{N_i}^i\}$ is a sequence of i.i.d. random variables, $\{Y_{1,D}^i, Y_{2,D}^i, \ldots, Y_{N_i,D}^i\}$ converges in distribution to $\{Y_1^i, Y_2^i, \ldots, Y_{N_i}^i\}$ when D is large, so that $Pr(L_{i,D} > u) = Pr(Y_{1,D}^i + \ldots + Y_{N_i,D}^i > u)$ converges to $Pr(Y_1^i + \ldots + Y_{N_i}^i > u) = \psi(u)$. Thus, $\psi_M(u)$ converges to $\psi(u)$ when n and D are large.

2.1 Numerical evaluation of the ruin probability approximation

To measure the performance of the proposed algorithm, we used a numerical evaluation. The claim amounts distributions *Expo*(1), *Expo*(2), *Gamma*(2,1), and *Gamma* (2,2) whose exact ruin probabilities are obtained by (4) and (5), were considered. We compared approximated values from the proposed method and the previously reported methods using their maximum absolute errors defined a s $\max|\psi(u) - \hat{\psi}(u)|$ for any approximation $\hat{\psi}(u)$. Moreover, we also compared all approximated values with the Lundberg upper bound of ruin probability in (7).

We set the security loading $\theta = 0.1, 0.3, 0.5,$ the intensity of the number of claims process $\lambda = 1,$ and the initial capital $u = 0, 5, 10, \ldots, 30.$ For the approximation based on simulation, we set the number of iterations $n = 500,000.$ For the proposed approximation, we set the number of truncated elements $D = 100.$ The comparisons are shown in Tables 1-4.

When the claim amounts distribution is exponential, $\psi_{DV}(u)$ and $\psi_B(u)$ are equal to the exact ruin probability $\psi(u)$. The results shown in Tables 1 and 2 show that $\psi_{PK}(u)$ and $\psi_M(u)$ were close to the exact ruin probability $\psi(u)$ and none of them were higher than e^{-Ru}. That means both of them gave reasonable values.

The maximum absolute error results in Tables 3 and 4 show that, when the claim amounts distribution was gamma, $\psi_M(u)$ performed the best in terms of overall deviation from the exact value. Moreover, all of the values of $\psi_M(u)$ were within e^{-Ru} whereas most values of $\psi_{PK}(u)$ were higher than e^{-Ru} when the security loading θ was large. The value of $\psi_{DV}(u)$ gave the highest maximum absolute error because there were more derivations between $\psi(0)$ and $\psi_{DV}(0)$. When the initial capital $u > 0$, both of $\psi_M(u)$ and $\psi_{DV}(u)$ gave similar results overall. However, in the case where the first three moments of claim amounts distribution did not exist, $\psi_{DV}(u)$ was undefined.

3. Minimum Initial Capital

The minimum initial capital is defined in Pairote *et al.* (2013) as follows:

Definition 1 Let $\{U(t), t \ge 0\}$ be a surplus process which is driven by the compound Poisson claim process $\{S(t), t \ge 0\}$ and $c > 0$ be the premium rate. For any $\alpha \in (0,1)$, let $u \ge 0$ be the initial capital. If $\psi(u) \le \alpha$, then u is called an acceptable initial capital level corresponding to $(\alpha, c, \{S(t), t \ge 0\})$. In particular, if

Table 1. Comparison of the approximate ruin probability with the exact ruin probability $\psi(u)$ and the Lundberg upper bound for claim amounts distribution *Expo* (1).

θ	R	u	$\psi(u)$	e^{-Ru}	$\psi_{PK}(u)$	$\psi_M(u)$
0.1	0.0909	0	0.9091	1.0000	0.9092	0.9092
		5	0.5770	0.6347	0.5774	0.5782
		10	0.3663	0.4029	0.3670	0.3659
		15	0.2325	0.2557	0.2326	0.2333
		20	0.1476	0.1623	0.1485	0.1472
		25	0.0937	0.1030	0.0936	0.0939
		30	0.0595	0.0654	0.0597	0.0594
0.3	0.2308	0	0.7692	1.0000	0.7691	0.7697
		5	0.2426	0.3154	0.2424	0.2423
		10	0.0765	0.0995	0.0764	0.0771
		15	0.0241	0.0314	0.0243	0.0239
		20	0.0076	0.0099	0.0075	0.0078
		25	0.0024	0.0031	0.0024	0.0024
		30	0.0008	0.0010	0.0008	0.0008
0.5	0.3333	0	0.6667	1.0000	0.6656	0.6677
		5	0.1259	0.1889	0.1262	0.1250
		10	0.0238	0.0357	0.0239	0.0240
		15	0.0045	0.0067	0.0045	0.0045
		20	0.0008	0.0013	0.0008	0.0009
		25	0.0002	0.0002	0.0002	0.0002
		30	0.0000	0.0000	0.0000	0.0000
Maximum Absolute Error					0.0011	0.0013

$\theta > 0.5$ is not considered because most $\psi(u)$ will be zero.

$$u_\alpha^* = \min_{u \ge 0}\{u : \psi(u) \le \alpha\} \tag{18}$$

exists, u_α^* is called the minimum initial capital corresponding to $(\alpha, c, \{S(t), t \ge 0\})$.

Lemma 1 Let non-ruin probability $\phi(u) = 1 - \psi(u)$, then the transformation through function $g(t)$ dened for $0 < t < 1$ by $g(t) = \phi^{-1}(t)$. We have the rst derivative of g as

$$g'(t) = \frac{d}{dt}\phi^{-1}(t) = \frac{1}{\phi'(\phi^{-1}(t))}.$$

Theorem 2 Let $\{L^1, L^2, \ldots\}$ be a sequence of maximal aggregate loss with each $L^j, j = 1, 2, \ldots, n$ being independent random variables and distributed according to non-ruin probability $\phi(u) = 1 - \psi(u)$. Let $L_{[n(1-\alpha)]}$ be the $n(1-\alpha)^{th}$ order statistic based on $\{L^1, L^2, \ldots, L^n\}$ and $\alpha \ge \theta/(1+\theta)$. Therefore,

$$\sqrt{n}\left(L_{[n(1-\alpha)]} - u_\alpha^*\right) \xrightarrow{d} W \sim N\left(0, \frac{\alpha(1-\alpha)}{\left[\phi'(u_\alpha^*)\right]^2}\right).$$

From our idea of proposed approximation of the ruin probability, we apply percentile estimation using Monte Carlo simulation to obtain the computed approximation of u_α^*. The algorithm is as follows:

1. Approximate F_X based on real data.
2. Set the numbers of iterations n, m, and number of truncated elements D to be some large numbers. Generate $N_i, i = 1, 2, \ldots, n$ from $Geometric(q)$, where $q = \theta/(1+\theta)$, and set them to be the number of new record highs.
3. Generate sequence $\{W_1, W_2, \cdots, W_D\}$ from i.i.d. $Exp(\lambda)$ and $\{X_1, X_2, \ldots, X_D\}$ with F_X. Let $T_j = W_1 + \ldots + W_j$ and $S_j = X_1 + \ldots + X_j$ be the timing of claims and values of the claims process, respectively. Compute the value of the process $\{S(t) - ct; 0 \le t \le T_D\}$ by $V_j = S_j - cT_j$, for $j = 1, 2, \ldots, D$.
4. If $V_j > 0$ for some $j = 1, 2, \ldots, D$, then let $Y_{1,D}^i$ be the first V_j above zero, else repeat step 3.
5. Obtain the amount of $\{Y_{2,D}^i, Y_{3,D}^i, \ldots, Y_{N_i,D}^i\}$ by repeating steps 2 to 4 and let $L_{i,D} = Y_{1,D}^i + \ldots + Y_{N_i,D}^i$.
6. Repeat steps 2 to 6 m times.

Table 2. Comparison of the approximate ruin probability with the exact ruin probability $\psi(u)$ and the Lundberg upper bound for claim amounts distribution *Expo* (2).

θ	R	u	$\psi(u)$	e^{-Ru}	$\psi_{PK}(u)$	$\psi_M(u)$
0.1	0.1818	0	0.9091	1.0000	0.9092	0.9095
		5	0.3663	0.4029	0.3672	0.3653
		10	0.1476	0.1623	0.1482	0.1478
		15	0.0595	0.0654	0.0592	0.0594
		20	0.0240	0.0263	0.0238	0.0239
		25	0.0097	0.0106	0.0097	0.0095
		30	0.0039	0.0043	0.0040	0.0040
0.3	0.4615	0	0.7692	1.0000	0.7691	0.7696
		5	0.0765	0.0995	0.0760	0.0764
		10	0.0076	0.0099	0.0076	0.0076
		15	0.0008	0.0010	0.0008	0.0008
		20	0.0001	0.0001	0.0001	0.0001
		25	0.0000	0.0000	0.0000	0.0000
		30	0.0000	0.0000	0.0000	0.0000
0.5	0.6667	0	0.6667	1.0000	0.6671	0.6665
		5	0.0238	0.0357	0.0236	0.0237
		10	0.0008	0.0013	0.0009	0.0008
		15	0.0000	0.0000	0.0000	0.0000
		20	0.0000	0.0000	0.0000	0.0000
		25	0.0000	0.0000	0.0000	0.0000
		30	0.0000	0.0000	0.0000	0.0000
Maximum Absolute Error					0.0009	0.0010

$\theta > 0.5$ is not considered because most $\psi(u)$ will be zero.

7. Let u_α^j be the $[m(1-\alpha)]^{th}$ smallest observation in $L_{N_1,D}, L_{N_2,D}, \ldots, L_{N_m,D}$.
8. Repeat steps 2 to 7 *n* times.
9. Estimate u_α^* by $\overline{u}_\alpha = \sum_{j=1}^{n} u_\alpha^j / n.$
10. Increase $n, m,$ and *D*, and repeat steps 2 to 9 until $\overline{u}_\alpha$ remains constant. From Theorems 1 and 2, $\overline{u}_\alpha$ converges to u_α^* as $n, m,$ and *D*, become large.

3.1 Performance evaluation of the proposed approximation

We evaluated the performance of the proposed approximation, $\overline{u}_\alpha$ it numerically. The claim amounts distributions $Expo(1)$, $Expo(2)$, $Gamma(2,1)$, and *Gamma* (2,2), that obtained the minimum initial capital u_α^* by setting (4) and (5) equal to α, were used. From the Lundberg upper bound (7) and the fact that the ruin probability is a non-increasing function in *u*, that means the upper bound of the minimum initial capital can derived as follows:

$$\psi(u_\alpha^*) = \alpha \quad \le e^{-Ru_\alpha^*},$$
$$u_\alpha^* \quad \le \frac{\ln \alpha}{R}. \tag{19}$$

If the simulation results show that the proposed estimate is greater than the upper bound for any cases, then the proposed estimator is considered not reasonable. We set the security loading $\theta = 0.1, 0.3, 0.5,$ the intensity of the number of claims process $\lambda = 1$, the acceptable level $\alpha = 0.05, 0.1, 0.2,$ the numbers of iterations $m = 5{,}000, n = 1{,}000$ and $D = 100$. The comparison of the proposed estimator $\overline{u}_\alpha$ and the upper bound of the minimum initial capital are shown in Table 5.

From Table 5, we can see that the proposed estimator $\overline{u}_\alpha$ was close to the exact minimum initial capital u_α^* when the claim amounts distribution was exponential or gamma. Moreover, all of the values $\overline{u}_\alpha$ were within the upper bound of the minimum initial capital. Thus, the proposed approximate $\overline{u}_\alpha$ is reasonable for the minimum initial capital.

Table 3. Comparison of the approximate ruin probability with the exact ruin probability $\psi(u)$ and the Lundberg upper bound for claim amounts distribution *Expo* (2,1).

θ	R	u	$\psi(u)$	e^{-Ru}	$\psi_{DV}(u)$	$\psi_{B}(u)$	$\psi_{PK}(u)$	$\psi_{M}(u)$
0.1	0.0613	0	0.9091	1.0000	0.9184	0.9091	0.9090	0.9091
		5	0.6767	0.7360	0.6762	0.6714	0.6237	0.6773
		10	0.4982	0.5417	0.4979	0.4959	0.4391	0.4991
		15	0.3668	0.3987	0.3666	0.3663	0.3224	0.3679
		20	0.2700	0.2935	0.2699	0.2705	0.2420	0.2703
		25	0.1988	0.2160	0.1987	0.1998	0.1849	0.1995
		30	0.1463	0.1590	0.1463	0.1476	0.1433	0.1467
0.3	0.1584	0	0.7692	1.0000	0.7895	0.7692	0.7702	0.7692
		5	0.3600	0.4529	0.3585	0.3564	0.3204	0.3600
		10	0.1631	0.2052	0.1628	0.1652	0.1511	0.1626
		15	0.0739	0.0929	0.0739	0.0765	0.0793	0.0736
		20	0.0335	0.0421	0.0336	0.0355	0.0448*	0.0332
		25	0.0152	0.0191	0.0152	0.0164	0.0258*	0.0154
		30	0.0069	0.0086	0.0069	0.0076	0.0159*	0.0069
0.5	0.2324	0	0.6667	1.0000	0.6923	0.6667	0.6680	0.6667
		5	0.2199	0.3129	0.2184	0.2195	0.1971	0.2191
		10	0.0688	0.0979	0.0689	0.0722	0.0710	0.0692
		15	0.0215	0.0306	0.0217	0.0238	0.0301	0.0213
		20	0.0067	0.0096	0.0069	0.0078	0.0142*	0.0067
		25	0.0021	0.0030	0.0022	0.0026	0.0070*	0.0020
		30	0.0007	0.0009	0.0007	0.0008	0.0038*	0.0006
Maximum Absolute Error					0.0256	0.0053	0.0591	0.0011

*the approximate exceeds the Lundberg upper bound

4. Application of the Approximation to Real Data

Data from a motor insurance company was used. When looking at the data from 2013, this company had an average number of motor insurance claims of 13.1275 cases per month. The number of claims distribution of this company was Poisson with parameter λ=13.1275. To obtain the claim amounts distribution that the company will face, 100 costs data form from five dealer garages were collected, and the claim amounts fitted to a Pareto distribution with the following density:

$$f_X(x) = \frac{\beta\delta^{\beta}}{x^{\beta+1}},\ x > \delta > 0,\ \beta > 0,$$

and $E\left[X^i\right] = \beta\delta^i / (\beta - i)$ where $\beta > i$. Two parameters of the Pareto distribution were estimated by the maximum likelihood estimator method and found to be $\beta = 0.6475$ and $\delta = 2.5680$. By Kolmogorov-Smirnov test, the Pareto claim amounts distribution was accepted with a p-value of 0.0592.

Since the first three moments of the claim amounts distribution are undefined, the De Vylder approximation and the Bowers approximation cannot be applied to this claim amounts data. However, we were able to use the proposed method since it does not require the second and third moments of the claim amounts distribution. In order to set a premium rate dependent on the expected value of claim amounts distribution, we used the sample mean of the claim amounts data to estimate it. The approximate ruin probability

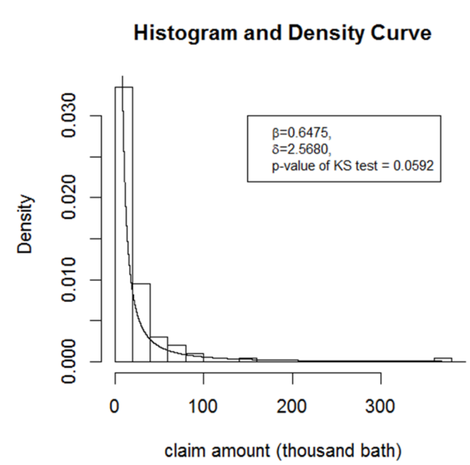

Figure 2. Claim modeling with 100 costs data from five dealer garages.

Table 4. Comparison of the approximate ruin probability with the exact ruin probability $\psi(u)$ and the Lundberg upper bound for claim amounts distribution *Expo* (2,2).

θ	R	u	$\psi(u)$	e^{-Ru}	$\psi_{DV}(u)$	$\psi_{B}(u)$	$\psi_{PK}(u)$	$\psi_{M}(u)$
0.1	0.1225	0	0.9091	1.0000	0.9184	0.9091	0.9077	0.9091
		5	0.4982	0.5420	0.4979	0.4959	0.4408	0.4986
		10	0.2700	0.2938	0.2699	0.2705	0.2408	0.2710
		15	0.1463	0.1592	0.1463	0.1476	0.1424	0.1472
		20	0.0793	0.0863	0.0793	0.0805	0.0889*	0.0800
		25	0.0430	0.0468	0.0430	0.0439	0.0574*	0.0432
		30	0.0233	0.0253	0.0233	0.0240	0.0381*	0.0236
0.3	0.3168	0	0.7692	1.0000	0.7895	0.7692	0.7687	0.7684
		5	0.1631	0.2052	0.1628	0.1652	0.1506	0.1638
		10	0.0335	0.0421	0.0336	0.0355	0.0444	0.0334
		15	0.0069	0.0086	0.0069	0.0076	0.0161*	0.0066
		20	0.0014	0.0018	0.0014	0.0016	0.0064*	0.0014
		25	0.0003	0.0004	0.0003	0.0004	0.0028*	0.0003
		30	0.0001	0.0001	0.0001	0.0001	0.0014*	0.0001
0.5	0.4648	0	0.6667	1.0000	0.6923	0.6667	0.6662	0.6663
		5	0.0688	0.0979	0.0689	0.0722	0.0713	0.0689
		10	0.0067	0.0096	0.0069	0.0078	0.0143*	0.0068
		15	0.0007	0.0009	0.0007	0.0008	0.0037*	0.0006
		20	0.0001	0.0001	0.0001	0.0001	0.0012*	0.0001
		25	0.0000	0.0000	0.0000	0.0000	0.0004*	0.0000
		30	0.0000	0.0000	0.0000	0.0000	0.0001*	0.0000
Maximum Absolute Error					0.0256	0.0034	0.0574	0.0010

*the approximate exceeds the Lundberg upper bound

and minimum initial capital for this company are shown in Tables 6 and 7, respectively.

5. Conclusions

When the claim amounts distribution is exponential or closely related to it, the ruin probability over infinite time with a classical continuous time surplus process exists. However, for other claim amounts distributions, the approximate ruin probability is used. In this study, we proposed a new simple approximate ruin probability for any claim amounts distribution. The numerical studies showed that almost all of the approximated ruin probability by the proposed approximation is reasonable and close to the exact ruin probability. In some situations, the proposed method gave better approximated values than other previously reported approximation methods.

By application of the proposed ruin probability approximation, we propose the approximate minimum initial capital for any claim amounts distribution. The numerical study showed that the proposed approximation was close to the exact minimum initial capital. Therefore, our proposed approximation is reasonable and useful for reserving the initial capital for managing the ruin probability of companies over innite time since it was not greater than the given quantity.

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Table 5. Comparison of the proposed estimator and the upper bound of the minimum initial capital when the claim amounts distribution is exponential and gamma.

θ	$F(x)$	α	u_α^*	$\overline{u}_\alpha$	$\ln\alpha / R$
0.1	*Expo* (1)	0.05	31.9046	31.5599	32.9531
	$R = 0.0909$	0.10	24.2800	24.6094	25.3284
		0.20	16.6554	16.0938	17.7038
	Expo (2)	0.05	15.9523	15.9683	16.4765
	$R = 0.1818$	0.10	12.1400	12.3779	12.6642
		0.20	8.3277	8.4085	8.8519
	Gamma (2,1)	0.05	47.5332	47.7969	48.9090
	$R = 0.0613$	0.10	36.2167	36.5589	37.5926
		0.20	24.9003	24.3273	26.2761
	Gamma (2,2)	0.05	23.7666	23.5672	24.4545
	$R = 0.1225$	0.10	18.1084	18.0899	18.7963
		0.20	12.4501	12.7812	13.1380
0.3	*Expo* (1)	0.05	11.8446	11.8493	12.9815
	$R = 0.2308$	0.10	8.8410	8.8602	9.9779
		0.20	5.8373	5.8541	6.9742
	Expo (2)	0.05	5.9223	5.8299	6.4908
	$R = 0.4615$	0.10	4.4205	4.3561	4.9889
		0.20	2.9187	2.9367	3.4871
	Gamma (2,1)	0.05	17.4632	17.5061	18.9140
	$R = 0.1584$	0.10	13.0869	13.0808	14.5377
		0.20	8.7106	8.9683	10.1614
	Gamma (2,2)	0.05	8.7316	8.6122	9.4570
	$R = 0.3168$	0.10	6.5435	6.6101	7.2689
		0.20	4.3553	4.4709	5.0807
0.5	*Expo* (1)	0.05	7.7708	7.9780	8.9881
	$R = 0.3333$	0.10	5.6914	5.5932	6.9084
		0.20	3.6119	3.5740	4.8288
	Expo (2)	0.05	3.8854	3.9890	4.4934
	$R = 0.6667$	0.10	2.8457	2.7966	3.4537
		0.20	1.8060	1.7870	2.4140
	Gamma (2,1)	0.05	11.3745	11.5783	12.8904
	$R = 0.2324$	0.10	8.3920	8.1980	9.9079
		0.20	5.4092	5.2827	6.9253
	Gamma (2,2)	0.05	5.6872	5.7891	6.4452
	$R = 0.4648$	0.10	4.1960	4.0990	4.9539
		0.20	2.7046	2.6413	3.4626

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Table 6. The approximate ruin probability with real data.

θ	u (thousand baht)	$\psi_M(u)$
0.1	0	0.9090
	10	0.8860
	20	0.8742
	30	0.8649
0.3	0	0.7677
	10	0.7273
	20	0.7073
	30	0.6922
0.5	0	0.6679
	10	0.6222
	20	0.5993
	30	0.5833

Table 7. The approximate minimum initial capital with real data.

θ	α	$\overline{u}_\alpha$ (thousand baht)
0.1	0.10	84,335.98
	0.20	27,070.93
	0.30	12,432.430
0.3	0.10	20,448.590
	0.20	6,129.938
	0.30	2,521.029
0.5	0.10	11,875.940
	0.20	3,380.349
	0.30	1,255.261

Appendix

Proof of Theorem 1. Let $F_{Y_{1,D}}$ and F_{Y_1} be distribution functions of $Y_{1,D}$ and Y_1 respectively, then

$$F_{Y_{1,D}}(y) = \Pr(Y_{1,D} < y) = \Pr\left(0 < S(t) - ct < y \middle| S(t) - ct > 0, \textit{for some } 0 < t \le T_D\right),$$

for some $y > 0$. When the number of truncated elements $D \to \infty$, it is obviously that the time at which the D^{th} claim occurred $T_D \to \infty$ as well. Thus

$$\begin{aligned}\lim_{D\to\infty} F_{Y_{1,D}}(y) &= \lim_{D\to\infty} \Pr\left(0 < S(t) - ct < y \middle| S(t) - ct > 0, \textit{for some } 0 < t \le T_D\right),\\ &= \Pr\left(0 < S(t) - ct < y \middle| S(t) - ct > 0, \textit{for some } 0 < t < \infty\right),\\ &= \Pr\left(0 < S(t) - ct < y \middle| S(t) - ct > 0, \textit{for some } t > 0\right),\\ &= F_{Y_1}(y).\end{aligned}$$

Proof of Lemma 1. Let $y = \phi^{-1}(t)$ iff $\phi(y) = t$, then $\phi'(y)\,dy = dt$ and $\dfrac{dy}{dt} = \dfrac{1}{\phi'(y)} = \dfrac{1}{\phi'\left(\phi^{-1}(t)\right)}$.

Proof of Theorem 2. Suppose the sequence of maximal aggregate loss consists of i.i.d. continuous random variables from a distribution with non-ruin probability ϕ. Let $\overline{Z}_n(u)$ be a random variable dened for positive initial capital u by

$$\overline{Z}_n(u) = \frac{1}{n}\sum_{i=1}^{n} Z_i(u),$$

where

$$Z_i(u) = \begin{cases} 1 & ; L^i \le u, \\ 0 & ; L^i > u. \end{cases},$$

Subsequently, $Z_i(u)$ has the expectation $E[Z_i(u)] = \Pr(L^i \le u) = \phi(u)$ and the variance $\sigma^2(u) = \phi(u)[1 - \phi(u)]$, and, by the central limit theorem,

$$\sqrt{n}\left(\bar{Z}_n(u)-\phi(u)\right)\xrightarrow{d}W\sim N\left(0,\phi(u)\left[1-\phi(u)\right]\right),$$

By Lemma 1, using the Delta method,

$$\sqrt{n}\left[\phi^{-1}\left(\bar{Z}_n(u)\right)-\phi^{-1}\left(\phi(u)\right)\right]\xrightarrow{d}W\sim N\left(0,\frac{\phi(u)\left[1-\phi(u)\right]}{\phi'\left(\phi^{-1}\left(\phi(u)\right)\right)}\right),$$

and, by replacing u by the minimum initial capital u_α^*, we have

$$\sqrt{n}\left[\phi^{-1}\left(\bar{Z}_n\left(u_\alpha^*\right)\right)-u_\alpha^*\right]\xrightarrow{d}W\sim N\left(0,\frac{\alpha(1-\alpha)}{\phi'\left(u_\alpha^*\right)}\right).$$

Now $\phi^{-1}\left(\bar{Z}_n(u)\right)$ is a random variable that lies between the order $\left[100(1-\alpha)-1\right]^{st}$ and $100(1-\alpha)^{th}$ sample quantile that can be written using order statistic notation as $L_{[n(1-\alpha)]}$. In fact,

$$\left|L_{[n(1-\alpha)]}-\phi^{-1}\left(\bar{Z}_n(u)\right)\right|\xrightarrow{a.s.}0.$$

It follows that

$$\sqrt{n}\left(L_{[n(1-\alpha)]}-u_\alpha^*\right)\xrightarrow{d}W\sim N\left(0,\frac{\alpha(1-\alpha)}{\left[\phi'\left(u_\alpha^*\right)\right]^2}\right),$$

and the proof of Theorem 2 is complete.