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Characterizations of rational numbers by SEL series and alternating SEL series expansions

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Abstract

We establish characterizations of rational numbers by using SEL series expansions, which yield generalized versions of characterizing rational numbers by Sylvester series, Engel series and Lüroth series expansions. Characterizations of rational numbers via alternating SEL series expansions yield generalized versions of characterizing rational numbers by alternating Sylvester series, alternating Engel series and alternating Lüroth series expansions.

Keywords: Sylvester and alternating Sylvester series, Engel and alternating Engel series, Lüroth and alternating Lüroth series, SEL and alternating SEL series

1. Introduction

According to Galambos (1976), Knopfmacher and Knopfmacher (1988), and Singthongla and Kanasri (2014), it is well-known that each $A \in \mathbb{R}$ is uniquely representable as an infinite series expansion called *Sylvester series expansion*, which is of the form

$$A = a_0 + \sum_{n=1}^{\infty} \frac{1}{a_n},$$

where $a_0 = \begin{cases} \lfloor A \rfloor & \text{if } A \notin \mathbb{Z} \\ A-1 & \text{if } A \in \mathbb{Z} \end{cases}, a_1 \geq 2,$ and $a_{n+1} \geq a_n(a_n - 1) + 1$ for all $n \geq 1$. Moreover, $A \in \mathbb{Q}$ if and only if $a_{n+1} = a_n(a_n - 1) + 1$ for all sufficiently large n. Ananalogous representation Galambos (1976), Knopfmacher and Knopfmacher (1988), Laohakosol, Chaichana, Kanasri, and Rattanamoong (2009), and Singthongla and Kanasri (2014) also states that every real number A has a unique representtation as an infinite series expansion called *Engel series expansion*, which is of the form

$$A = a_0 + \sum_{n=1}^{\infty} \frac{1}{a_1 \cdots a_n},$$

where $a_0 = \begin{cases} \lfloor A \rfloor & \text{if } A \notin \mathbb{Z} \\ A-1 & \text{if } A \in \mathbb{Z} \end{cases}, a_1 \geq 2,$ and $a_{n+1} \geq a_n$ for all $n \geq 1$. Moreover, $A \in \mathbb{Q}$ if and only if $a_{n+1} = a_n$ for all sufficiently large n. For the last representation Galambos (1976) and Singthongla and Kanasri (2014), it is also known that each $A \in \mathbb{R}$ is uniquely representable as an infinite

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series expansion called *Lüroth series expansion*, which is of the form

$$A = a_0 + \frac{1}{a_1} + \sum_{n=1}^{\infty} \frac{1}{a_1(a_1-1)\cdots a_n(a_n-1)a_{n+1}},$$

where $a_0 = \begin{cases} \lfloor A \rfloor & \text{if } A \notin \mathbb{Z} \\ A-1 & \text{if } A \in \mathbb{Z} \end{cases}$ and $a_n \geq 2$ for all $n \geq 1$. Moreover, $A \in \mathbb{Q}$ if and only if the Lüroth series expansion of A is periodic.

In another direction, Kalpazidou, Knopfmacher, and Knopfmacher introduced three alternating series expansions for real numbers, namely, alternating Sylvester series, alternating Engel series (Knopfmacher & Knopfmacher, 1989), and alternating Lüroth series expansions (Kalpazidou, Knopfmacher, & Knopfmacher, 1990). The series under discussion are as follows: Every real number A has a unique representation as a series expansion called *alternating Sylvester series expansion*, which is of the form

$$A = a_0 + \frac{1}{a_1} + \sum_{n=1}^{\infty} \frac{(-1)^n}{a_{n+1}},$$

where $a_0 = \lfloor A \rfloor$, $a_1 \geq 1$, and $a_{n+1} \geq a_n(a_n+1)$ for all $n \geq 1$. Moreover, A is rational if and only if the alternating Sylvester series expansion of A is finite. Corresponding to the series of Engel, it is known that every real number A has a unique representation as a series expansion called *alternating Engel series expansion*, which is of the form

$$A = a_0 + \frac{1}{a_1} + \sum_{n=1}^{\infty} \frac{(-1)^n}{a_1 \cdots a_n a_{n+1}},$$

where $a_0 = \lfloor A \rfloor$, $a_1 \geq 1$, and $a_{n+1} \geq a_n + 1$ for all $n \geq 1$. Moreover, A is rational if and only if the alternating Engel series expansion of A is finite. For the last representation, it is also known that every real number A has a unique representation as a series expansion called *alternating Lüroth series expansion*, which is of the form

$$A = a_0 + \frac{1}{a_1} + \sum_{n=1}^{\infty} \frac{(-1)^n}{a_1(a_1+1)\cdots a_n(a_n+1)a_{n+1}},$$

where $a_0 = \lfloor A \rfloor$ and $a_n \geq 1$ for all $n \geq 1$. Moreover, A is rational if and only if the alternating Lüroth series expansion of A is finite or periodic.

In 2014, Kanasri and Singthongla introduced two algorithms for constructing two infinite series expansions for real numbers, namely, *SEL series expansion* (Singthongla & Kanasri, 2014) and *alternating SEL series expansion* (Kanasri & Singthongla, 2014), which yield generalized versions of the first three series expansions and the last three (alternating) series expansions, respectively.

In this work, we establish characterizations of rational numbers by using SEL series expansion and alternating SEL series expansion, which yield generalized versions of characterizing rational numbers by the first three series expansions and the last three (alternating) series expansions, respectively.

2. Characterizing Rational Numbers by SEL Series Expansion

We first recall the algorithm for constructing SEL Series Expansion as follows: Given any real number A, write it as $A = a_0 + A_1$, where $a_0 = \begin{cases} \lfloor A \rfloor & \text{if } A \notin \mathbb{Z} \\ A-1 & \text{if } A \in \mathbb{Z} \end{cases}$

and $0 < A_1 \leq 1$. Then recursively define

$$a_n = 1 + \left\lfloor \frac{1}{A_n} \right\rfloor, \tag{2.1}$$

$$A_{n+1} = (a_n A_n - 1)e_n, \tag{2.2}$$

where $e_n = e_n(a_n)$ is a positive rational number, which may depend on a_n, for all $n \geq 1$.

To facilitate the proof of Theorem 1, we prove the following lemma.

Lemma 1. *Any series*

$$\frac{1}{b_1} + \sum_{n=1}^{\infty} \frac{1}{b_1 f_1 \cdots b_n f_n b_{n+1}}, \tag{2.3}$$

where $b_n \in \mathbb{N}, b_1 \geq 2,$ $b_{n+1} \geq (b_n - 1)/f_n + 1 \geq 2,$ *and* $f_n = f_n(b_n) \in \mathbb{Q}^+$ *for all* $n \geq 1$ *converges to a real number* B_1 *such that* $b_1 = 1 + \lfloor 1/B_1 \rfloor$.

Proof. For $n \in \mathbb{N}$, let $C_n = \frac{1}{b_1} + \sum_{k=1}^{n-1} \frac{1}{b_1 f_1 \cdots b_k f_k b_{k+1}}$. It is clear that the sequence of positive real numbers (C_n) is increasing. To show that the series in (2.3) converges, it suffices to show that (C_n) is bounded above. Since $b_1 \geq 2$ and $b_{n+1} \geq (b_n - 1)/f_n + 1 \geq 2$ for all $n \geq 1$, we have

$$\frac{1}{f_n} \leq \frac{b_{n+1} - 1}{b_n - 1} (n \geq 1). \tag{2.4}$$

It follows that

$$\begin{aligned}
C_n &\leq \frac{1}{b_1} + \sum_{k=1}^{n-1} \frac{(b_2 - 1)(b_3 - 1)}{(b_1 - 1)(b_2 - 1)} \cdots \frac{(b_{k+1} - 1)}{(b_k - 1)} \cdot \frac{1}{b_1 \cdots b_k b_{k+1}} \\
&= \frac{1}{b_1} + \sum_{k=1}^{n-1} \frac{(b_{k+1} - 1)}{(b_1 - 1) b_1 \cdots b_k b_{k+1}} \\
&= \frac{(b_1 - 1)}{(b_1 - 1) b_1} + \frac{1}{b_1 - 1} \sum_{k=1}^{n-1} \left(\frac{1}{b_1 \cdots b_k} - \frac{1}{b_1 \cdots b_k b_{k+1}} \right) \\
&= \frac{1}{b_1 - 1} \left(1 - \frac{1}{b_1} + \frac{1}{b_1} - \frac{1}{b_1 \cdots b_n} \right) \\
&= \frac{1}{b_1 - 1} \left(1 - \frac{1}{b_1 \cdots b_n} \right) < \frac{1}{b_1 - 1},
\end{aligned}$$

for all $n \geq 1$. Thus the series in (2.3) converges to a nonzero real number B_1. Consequently, $1/b_1 < B_1 \leq 1/(b_1 - 1)$, which implies that $b_1 = 1 + \lfloor 1/B_1 \rfloor$, and the lemma follows.

Recall the following result in (Singthongla & Kanasri, 2014), which gives the existence and uniqueness of SEL series expansion for any real number.

Theorem 1. *Let* $A \in \mathbb{R}$ *and assume that*

$$\frac{a_n - 1}{e_n} \in \mathbb{N} \quad (n \geq 1). \tag{2.5}$$

Then A is uniquely representable as an infinite series expansion called SEL series expansion, which is of the form

$$A = a_0 + \frac{1}{a_1} + \sum_{n=1}^{\infty} \frac{1}{a_1 e_1 \cdots a_n e_n a_{n+1}}, \tag{2.6}$$

where $a_1 \geq 2$ *and* $a_{n+1} \geq (a_n - 1)/e_n + 1$ *for all* $n \geq 1$.

Proof. Using (2.1), we get

$$\frac{1}{a_n} < A_n \leq \frac{1}{a_n - 1} (n \geq 1). \tag{2.7}$$

We now prove the following Claim.

Claim. $0 < A_n \leq 1$ for all $n \geq 1$.

Proof of the Claim. We will prove this claim by induction on n. If $n = 1$, then we have seen that $0 < A_1 \leq 1$. Assume now that $0 < A_n \leq 1$ for $n \geq 1$. It follows by (2.1) that $a_n \geq 2$. Since $A_{n+1} = (a_n A_n - 1)e_n = \left(A_n - \frac{1}{a_n}\right) a_n e_n$ and using (2.5) and (2.7) we have

$$0 < A_{n+1} = (a_n A_n - 1)e_n \leq \left(\frac{a_n}{a_n - 1} - 1\right) e_n = \frac{e_n}{a_n - 1} \leq 1 \tag{2.8}$$

and so we have the Claim.

From this Claim together with (2.1), we deduce that $a_n \geq 2$ for all $n \geq 1$. By using (2.1), (2.5), and (2.8), we obtain $a_{n+1} \geq (a_n - 1)/e_n + 1$ for all $n \geq 1$.

From (2.2), we get that

$$A_n = \frac{1}{a_n} + \frac{A_{n+1}}{a_n e_n} (n \geq 1). \tag{2.9}$$

Applying (2.9) repeatedly, we obtain

$$A_1 = \frac{1}{a_1} + \frac{1}{a_1 e_1 a_2} + \cdots + \frac{1}{a_1 e_1 \cdots a_{n-1} e_{n-1} a_n} + \frac{A_{n+1}}{a_1 e_1 \cdots a_n e_n} (n \geq 1).$$

By Lemma 1, the series in the right hand side of (2.6) is convergent. It follows that $\lim_{n\to\infty} 1/(a_1 e_1 \cdots a_n e_n a_{n+1}) = 0$. Using (2.7) and $a_{n+1} \geq 2$, we deduce that

$$0 < \frac{A_{n+1}}{a_1 e_1 \cdots a_n e_n} \leq \frac{1}{a_1 e_1 \cdots a_n e_n} \cdot \frac{1}{a_{n+1} - 1} \leq \frac{2}{a_1 e_1 \cdots a_n e_n a_{n+1}} (n \geq 1).$$

It follows that $\lim_{n\to\infty} A_{n+1}/(a_1 e_1 \cdots a_{n-1} e_{n-1} a_n e_n) = 0$. Therefore,

$$A_1 = \frac{1}{a_1} + \sum_{n=1}^{\infty} \frac{1}{a_1 e_1 \cdots a_n e_n a_{n+1}}$$

and (2.6) follows, as desired.

To prove uniqueness, suppose that $A \in \mathbb{R}$ has expansions

$$A = a_0 + \frac{1}{a_1} + \sum_{n=1}^{\infty} \frac{1}{a_1 e_1 \cdots a_n e_n a_{n+1}} = b_0 + \frac{1}{b_1} + \sum_{n=1}^{\infty} \frac{1}{b_1 f_1 \cdots b_n f_n b_{n+1}}, \tag{2.10}$$

with the restrictions

$$a_0 \in \mathbb{Z}, a_n \in \mathbb{N}, a_1 \geq 2, a_{n+1} \geq \frac{a_n - 1}{e_n} + 1 \geq 2, \text{and} e_n = e_n(a_n) \in \mathbb{Q}^+ (n \geq 1),$$

$$b_0 \in \mathbb{Z}, b_n \in \mathbb{N}, b_1 \geq 2, b_{n+1} \geq \frac{b_n - 1}{f_n} + 1 \geq 2, \text{and} f_n = f_n(b_n) \in \mathbb{Q}^+ (n \geq 1).$$

Using Lemma 1, we obtain

$$0 < \frac{1}{a_1} < A_1 := \frac{1}{a_1} + \sum_{n=1}^{\infty} \frac{1}{a_1 e_1 \cdots a_n e_n a_{n+1}} \leq \frac{1}{a_1 - 1} \leq 1,$$

$$0 < \frac{1}{b_1} < B_1 := \frac{1}{b_1} + \sum_{n=1}^{\infty} \frac{1}{b_1 f_1 \cdots b_n f_n b_{n+1}} \leq \frac{1}{b_1 - 1} \leq 1.$$

If $A_1 = 1$, then by (2.10) we also have $B_1 = 1$, forcing $a_0 = b_0$. If $0 < A_1 < 1$, then (2.10) shows that $0 < B_1 < 1$, again forcing $a_0 = b_0$. In either case, by cancelling the terms a_0, b_0 in (2.10), we have

$$A_1 = \frac{1}{a_1} + \sum_{n=1}^{\infty} \frac{1}{a_1 e_1 \cdots a_n e_n a_{n+1}} = \frac{1}{b_1} + \sum_{n=1}^{\infty} \frac{1}{b_1 f_1 \cdots b_n f_n b_{n+1}} = B_1. \tag{2.11}$$

By Lemma 1, we deduce that $a_1 = 1 + \left\lfloor \frac{1}{A_1} \right\rfloor = 1 + \left\lfloor \frac{1}{B_1} \right\rfloor = b_1$, and so $e_1 = e_1(a_1) = e_1(b_1) = f_1$. By cancelling the terms a_1, b_1, and e_1, f_1 in (2.11), we obtain

$$A_2 := \frac{1}{a_2} + \sum_{n=2}^{\infty} \frac{1}{a_2 e_2 \cdots a_n e_n a_{n+1}} = \frac{1}{b_2} + \sum_{n=2}^{\infty} \frac{1}{b_2 f_2 \cdots b_n f_n b_{n+1}} =: B_2.$$

Then Lemma 1 implies that $a_2 = 1 + \left\lfloor \frac{1}{A_2} \right\rfloor = 1 + \left\lfloor \frac{1}{B_2} \right\rfloor = b_2$, and so $e_2 = e_2(a_2) = e_2(b_2) = f_2$. On repeating this argument, we successively find that $a_n = b_n$ and $e_n = f_n$ for all $n \geq 1$. Therefore, the expansion is unique and the proof of the theorem is complete.

By setting $e_n = 1/a_n$, $e_n = 1$, or $e_n = a_n - 1$ for all $n \geq 1$ in Theorem 1, we obtain the well-known expansions for real numbers, namely, Sylvester series, Engel series, or Lüroth series expansions, respectively.

In the first part of this work, we are interested in characterizing rational numbers by using SEL series expansion. First, we will give the definition of periodic SEL series expansion as follows: An SEL series expansion of a real number A,

$$a_0 + \frac{1}{a_1} + \sum_{n=1}^{\infty} \frac{1}{a_1 e_1 \cdots a_n e_n a_{n+1}},$$

is said to be *periodic* if there are positive integers m and r such that $a_n = a_{n+r}$ and $e_n = e_{n+r}$ for every $n \geq m$.

The following two theorems give characterizations of rational numbers by such expansion, which are our first main results.

Theorem 2. *Assume that* $1/e_n \in \mathbb{N}$ *for all* $n \geq 1$ *and let*

$$A = a_0 + \frac{1}{a_1} + \sum_{n=1}^{\infty} \frac{1}{a_1 e_1 \cdots a_n e_n a_{n+1}}$$

be the SEL series expansion of $A \in \mathbb{R}$. *Then* $A \in \mathbb{Q}$ *if and only if* $a_{n+1} = (a_n - 1)/e_n + 1$ *for all sufficiently large* n.

Proof. Assume that $A \in \mathbb{Q}$. Thus, by (2.2), each A_n is also rational, i.e., $A_n = p_n/q_n$, where p_n and q_n are positive integers with $(p_n, q_n) = 1$. Replacing A_n by p_n/q_n in (2.2) and (2.7), we obtain

$$\frac{a_n p_n - q_n}{(1/e_n) q_n} = \frac{p_{n+1}}{q_{n+1}} \tag{2.12}$$

and $a_n p_n - q_n \le p_n$, respectively, for all $n \ge 1$. Since $1/e_n \in \mathbb{N}$ and $(p_{n+1}, q_{n+1}) = 1$, we have

$$p_{n+1} \le a_n p_n - q_n \le p_n (n \ge 1). \tag{2.13}$$

It follows that $p_n = P \ge 1$ for all sufficiently large n. By (2.13), we have $a_n P - q_n = P$ for all sufficiently large n. Consequently, P divides q_n, which implies $P = 1$, and so $q_n = a_n - 1$ for all sufficiently large n. Using (2.12), we deduce that

$$a_{n+1} - 1 = q_{n+1} = \frac{1}{e_n} q_n = \frac{a_n - 1}{e_n}$$

for all sufficiently large n.

Conversely, assume that there exists a positive integer N such that $a_{n+1} = (a_n - 1)/e_n + 1$ for all $n \ge N$. Then

$$\frac{a_{n+1} - 1}{a_n - 1} = \frac{1}{e_n} (n \ge N). \tag{2.14}$$

It follows that

$$\begin{aligned}
A &= B + \frac{1}{a_1 e_1 \cdots a_{N-1} e_{N-1}} \sum_{n=N}^{\infty} \frac{1}{a_N e_N \cdots a_n e_n a_{n+1}} \\
&= B + \frac{1}{\alpha} \sum_{n=N}^{\infty} \frac{1}{a_N a_{N+1} \cdots a_n a_{n+1}} \left(\frac{a_{N+1} - 1}{a_N - 1}\right)\left(\frac{a_{N+2} - 1}{a_{N+1} - 1}\right) \cdots \left(\frac{a_{n+1} - 1}{a_n - 1}\right) \\
&= B + \frac{1}{\alpha} \sum_{n=N}^{\infty} \frac{a_{n+1} - 1}{(a_N - 1) a_N a_{N+1} \cdots a_n a_{n+1}} \\
&= B + \frac{1}{\alpha (a_N - 1)} \sum_{n=N}^{\infty} \left(\frac{1}{a_N a_{N+1} \cdots a_n} - \frac{1}{a_N a_{N+1} \cdots a_{n+1}}\right) \\
&= B + \frac{1}{\alpha (a_N - 1)} \left(\frac{1}{a_N}\right) \in \mathbb{Q},
\end{aligned}$$

where $B = a_0 + 1/a_1 + \sum_{n=1}^{N-1} 1/(a_1 e_1 \cdots a_n e_n a_{n+1})$ and $\alpha = a_1 e_1 \cdots a_{N-1} e_{N-1}$, as desired.

Theorem 3. *If $e_n \in \mathbb{N}$ for all $n \ge 1$, then the corresponding SEL series expansion of $A \in \mathbb{R}$ is periodic if and only if $A \in \mathbb{Q}$.*

Proof. Assume that the corresponding SEL series expansion of $A \in \mathbb{R}$ is periodic. Then there are positive integers m and r such that $a_n = a_{n+r}$ and $e_n = e_{n+r}$ for every $n \ge m$. Thus

$$\begin{aligned}
A &= a_0 + \frac{1}{a_1} + \cdots + \frac{1}{a_1 e_1 \cdots a_{m-2} e_{m-2} a_{m-1}} + \sum_{n=m}^{\infty} \frac{1}{a_1 e_1 \cdots a_{n-1} e_{n-1} a_n} \\
&= B + \frac{1}{a_1 e_1 \cdots a_{m-1} e_{m-1}} (\alpha + \beta\alpha + \beta^2 \alpha + \cdots) \\
&= B + \frac{\alpha}{a_1 e_1 \cdots a_{m-1} e_{m-1}} \left(\frac{1}{1 - \beta}\right) \in \mathbb{Q},
\end{aligned}$$

as desired, where $B \coloneqq a_0 + \frac{1}{a_1} + \cdots + \frac{1}{a_1 e_1 \cdots a_{m-2} e_{m-2} a_{m-1}}$,

$\alpha \coloneqq \frac{1}{a_m} + \cdots + \frac{1}{a_m e_m \cdots a_{m+r-1}}$ and $0 < \beta \coloneqq \frac{1}{a_m e_m \cdots a_{m+r-1} e_{m+r-1}} < 1$.

Conversely, we will show that the SEL series expansion of a rational number A is periodic. Since A is a rational, so is each A_n for all $n \geq 1$. Thus $A_n = p_n/q_n$, where p_n and q_n are positive integers with $(p_n, q_n) = 1$. Replacing A_n by p_n/q_n in (2.2), we get

$$\frac{e_n(a_n p_n - q_n)}{q_n} = \frac{p_{n+1}}{q_{n+1}} (n \geq 1).$$

Since $e_n \in \mathbb{N}$ and $(p_{n+1}, q_{n+1}) = 1$, we have $p_{n+1} \leq e_n(a_n p_n - q_n)$ and $1 \leq q_{n+1} \leq q_n$ for all $n \geq 1$. Then there exists a positive integer n_0 such that $q_n = Q$ for all $n \geq n_0$. Using (2.7), we get that $p_n(a_n - 1) \leq q_n$ for all $n \geq 1$. Thus

$$0 < p_n \leq p_n(a_n - 1) \leq Q \qquad (n \geq n_0).$$

But the number of positive integers in the interval $(0, Q]$ is finite implying that there are positive integers m $(\geq n_0)$ and r such that $p_{m+r} = p_m$ and so $A_{m+r} = A_m$. Thus, by (2.1), we have $a_{m+r} = a_m$ and the assertion follows.

Characterizations of rational numbers using Sylvester series or Engel series expansions follow immediately from Theorem 2 by setting $e_n = 1/a_n$ or $e_n = 1$ for all $n \geq 1$, respectively. Taking $e_n = a_n - 1$ for all $n \geq 1$ in Theorem 3 leads to a characterization of rational numbers by Lüroth series expansion.

3. Characterizing Rational Numbers by Alternating SEL Series Expansion

In this section, we will establish characterizations of rational numbers by using alternating SEL series expansion. The periodic alternating SEL series expansion is defined similarly to the periodic SEL series expansion. We first recall the algorithm for constructing such expansion. Given any real number A, write it as $A = a_0 + A_1$, where $a_0 = \lfloor A \rfloor$ and $0 \leq A_1 < 1$. Then we recursively define

$$a_n = \left\lfloor \frac{1}{A_n} \right\rfloor, \quad \text{for} A_n > 0, \tag{3.1}$$

$$A_{n+1} = (1 - a_n A_n) e_n, \tag{3.2}$$

where $e_n = e_n(a_n)$ is a positive rational number, which may depend on a_n. By the proof of Theorem 1 in Kanasri and Singthongla (2014), we have the following facts on the alternating SEL series expansion:

(F1) $1/(a_n + 1) < A_n \leq 1/a_n$ for $n \geq 1, A_n > 0$,

(F2) for $n \geq 1$, if $0 < A_n < 1$, then $0 \leq A_{n+1} < 1$, and

(F3) if $A_m = 0$ for some $m \geq 1$, then the corresponding alternating SEL series expansion of A is finite.

The following theorem gives the existence and uniqueness of alternating SEL series expansion for any real number (Kanasri & Singthongla, 2014).

Theorem 4. *Let A be any real number and assume that $(a_n + 1)/e_n \in \mathbb{N}$ for all $n \geq 1$. Then A is uniquely representable as a series expansion called alternating SEL series expansion, which is of the form*

$$A = a_0 + \frac{1}{a_1} + \sum_{n=1}^{\infty} \frac{(-1)^n}{a_1 e_1 \cdots a_n e_n a_{n+1}},$$

where $a_1 \geq 1$ and $a_{n+1} \geq (a_n + 1)/e_n$ for all $n \geq 1$.

We deduce now our results on the alternating SEL series expansion for rational numbers.

Theorem 5. *If* $1/e_n \in \mathbb{N}$ *for all* $n \geq 1$, *then the corresponding alternating SEL series expansion of* $A \in \mathbb{R}$ *is finite if and only if* $A \in \mathbb{Q}$.

Proof. It is clear that any number represented by a finite alternating SEL series expansion is rational.

Conversely, assume that A is rational. Then, by (3.2), A_n is also rational for all $n \geq 1$. Thus, we can write $A_n = p_n/q_n$, where p_n and q_n are non-negative integers with $(p_n, q_n) = 1$ for all $n \geq 1$. By (3.1), we have $a_n = \lfloor 1/A_n \rfloor > 1/A_n - 1$ and so $q_n - a_n p_n < p_n$ for all $n \geq 1$. Replacing A_n by p_n/q_n in (3.2), we obtain

$$\frac{q_n p_{n+1}}{e_n} = (q_n - a_n p_n) q_{n+1} (n \geq 1).$$

Since $1/e_n \in \mathbb{N}$ and $(p_{n+1}, q_{n+1}) = 1$, we have $p_{n+1} | (q_n - a_n p_n)$ and so

$$0 \leq p_{n+1} \leq (q_n - a_n p_n) < p_n (n \geq 1).$$

This shows that (p_n) is a strictly decreasing sequence of non-negative integers. It follows that $p_m = 0$ for some $m \in \mathbb{N}$ and so $A_m = 0$. Thus, the assertion follows by (F3).

Theorem 6. *If* $e_n \in \mathbb{N}$ *for all* $n \geq 1$, *then the corresponding alternating SEL series expansion of* $A \in \mathbb{R}$ *is finite or periodic if and only if* $A \in \mathbb{Q}$.

Proof. If the corresponding alternating SEL series expansion of $A \in \mathbb{R}$ is finite, then it is clear that A is rational. Now we assume that such expansion is periodic. Then there are positive integers m and r such that $a_n = a_{n+r}$ and $e_n = e_{n+r}$ for every $n \geq m$. Let $\alpha = a_m e_m \cdots a_{m+r-1} e_{m+r-1}$ and $\beta = a_1 e_1 \cdots a_{m-1} e_{m-1}$. If $\alpha = 1$, then $a_m = a_{m+1} = \cdots = a_{m+r-1} = 1$ and $e_m = e_{m+1} = \cdots = e_{m+r-1} = 1$. But we have $a_{m+1} \geq (a_m + 1)/e_m = a_m + 1 > a_m$, a contradiction, so $\alpha > 1$. As the expansion is periodic, we derive

$$A = B + \frac{(-1)^{m-1} C}{\beta} \left(1 + \frac{(-1)^r}{\alpha} + \frac{(-1)^{2r}}{\alpha^2} + \cdots \right),$$

where

$$B = a_0 + \frac{1}{a_1} - \frac{1}{a_1 e_1 a_2} + \cdots + \frac{(-1)^{m-2}}{a_1 e_1 \cdots a_{m-2} e_{m-2} a_{m-1}},$$
$$C = \frac{1}{a_m} - \frac{1}{a_m e_m a_{m+1}} + \cdots + \frac{(-1)^{r-1}}{a_m e_m \cdots a_{m+r-2} e_{m+r-2} a_{m+r-1}}.$$

If r is even, then

$$A = B + \frac{(-1)^{m-1} C}{\beta} \left(1 + \frac{1}{\alpha} + \frac{1}{\alpha^2} + \cdots \right) = B + \frac{(-1)^{m-1} C \alpha}{\beta(\alpha - 1)} \in \mathbb{Q}.$$

Similarly, if r is odd, then

$$\begin{aligned} A &= B + \frac{(-1)^{m-1} C}{\beta} \left(1 - \frac{1}{\alpha} + \frac{1}{\alpha^2} - \cdots \right) \\ &= B + \frac{(-1)^{m-1} C}{\beta} \left\{ \left(1 + \frac{1}{\alpha^2} + \frac{1}{\alpha^4} + \cdots \right) - \left(\frac{1}{\alpha} + \frac{1}{\alpha^3} + \frac{1}{\alpha^5} + \cdots \right) \right\} \\ &= B + \frac{(-1)^{m-1} C \alpha}{\beta(\alpha + 1)} \in \mathbb{Q}. \end{aligned}$$

Conversely, assume that A is rational and the corresponding alternating SEL series expansion of $A \in \mathbb{R}$ is infinite. Then, by using (3.2), A_n is also rational for all $n \geq 1$. Let $A_1 = p/q$ for some positive integers p and q. Thus, for $n \geq 2$,

$$
\begin{aligned}
A_n &= (1 - a_{n-1}A_{n-1})e_{n-1} \\
&= e_{n-1} - e_{n-1}a_{n-1}A_{n-1} \\
&= e_{n-1} - e_{n-1}a_{n-1}(e_{n-2} - e_{n-2}a_{n-2}A_{n-2}) \\
&= e_{n-1} - e_{n-1}a_{n-1}e_{n-2} + e_{n-1}a_{n-1}e_{n-2}a_{n-2}A_{n-2} \\
&\vdots \\
&= aA_1 + b = \frac{p_n}{q},
\end{aligned}
$$

for some $a, b, p_n \in \mathbb{Z}$, since $e_n \in \mathbb{N}$ for all $n \geq 1$. By (F2) and (F3), we deduce that $0 < A_n < 1$ for all $n \geq 1$. It follows that $1 \leq p_n < q$ for all $n \geq 1$ i.e.

$$A_n \in \left\{\frac{1}{q}, \frac{2}{q}, \dots, \frac{q-1}{q}\right\} (n \geq 1),$$

which implies that the expansion is periodic as desired.

Characterizations of rational numbers using alternating Sylvester series or alternating Engel series follow immediately form Theorem 5 by setting $e_n = 1/a_n$ or $e_n = 1$ for all $n \geq 1$, respectively. Taking $e_n = a_n + 1$ for all $n \geq 1$ in Theorem 6 leads to a characterization of rational numbers by alternating Lüroth series expansion.

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