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Original Article

Monoid of linear hypersubstitutions for algebraic systems of type $((n),(2))$ and its regularity

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Abstract

An algebraic system is a structure which consists of a nonempty set together with a sequence of operations and a sequence of relations on this set. Properties of this structure are expressed in terms and formulas. In this paper, we show that the set of all linear hypersubstitutions for algebraic systems of the type $((n),(2))$ with a binary operation on this set and the identity element forms a monoid. Finally, we characterize idempotent and regular elements on the monoid.

Keywords: algebraic systems, linear terms, linear formulas, linear hypersubstitutions

1. Introduction

The concept of an algebraic system was first introduced by A.I. Mal'cev in 1973. For approach to algebraic systems, we need some preparations. Let A be a nonempty set and $n \in \mathbb{N}^+ := \mathbb{N} \setminus \{0\}$. An n-ary operation on A is a mapping $f: A^n \to A$. We call n the arity of f. An n-ary relation on A is a relation $\gamma \subseteq A^n$ and call n the arity of γ. Let I, J be indexed sets and let $(f_i)_{i\in I}, (\gamma_j)_{j\in J}$ be sequences of operation symbols and relation symbols, respectively. Let $\tau = (n_i)_{i\in I}$ and $\acute{\tau} = (n_j)_{j\in J}$ where f_i has the arity n_i for every $i \in I$ and γ_j has the arity n_j for every $j \in J$.

Definition 1.1 (Mal'cev, 1973) An algebraic system of type $(\tau, \acute{\tau})$ is a triple $\mathcal{A} := (A, (f_i)_{i\in I}, (\gamma_j)_{j\in J})$ consisting of a nonempty set A, a sequence $(f_i)_{i\in I}$ of operations on A where f_i is n_i-ary for $i \in I$ and a sequence $(\gamma_j)_{j\in J}$ of relations on A where γ_j is n_j-ary for $j \in J$. The pair $(\tau, \acute{\tau})$ is called the type of an algebraic system.

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To classify algebras into collections called varieties we need terms and some pairs of terms, i.e. equations. To classify algebraic systems into subclasses by logical sentences we need a language, i.e. quantifier free formulas.

Now, we recall basic notions related to terms. For a natural number $n \geq 1$, let $X_n = \{x_1, \dots, x_n\}$ be a finite set of variables, and let $X := \bigcup_{n\geq 1} X_n = \{x_1, \dots, x_n, \dots\}$ be countably infinite. Let $\{f_i | i \in I\}$ be a set of operation symbols which is disjoint from X. An n-ary term of type τ is defined inductively as follows:

(i) Every variable $x_j \in X_n$ is an n-ary term of type τ.

(ii) If $t_1, \dots, t_{n_i}$ are n-ary terms of type τ and f_i is an n_i-ary operation symbol, then $f_i(t_1, \dots, t_{n_i})$ is an n-ary term of type τ.

Let $W_\tau(X_n)$ be the set of all n-ary terms of type τ which contains $x_1, \dots, x_n$ and is closed under finite application of (ii).

Not all of the terms in the second-order language will used to express properties of algebraic systems. The one is called formulas, first introduced by A.I. Mal'cev in 1973. For approach to formulas see also (Mal'cev, 1973), and we recall the definition of formula which is defined by Denecke and Phusanga (2008).To define the quantifier free formulas of type $(\tau,\acute{\tau})$we need the logical connectives $\neg$ (negation), $\vee$ (disjunction) and the equation symbol $\approx$.

Definition 1.2. (Denecke & Phusanga, 2008) Let $n \in \mathbb{N}^+$. An n-ary quantifier free formula of type $(\tau,\acute{\tau})$(for simply, formula) is defined in the following steps:

(i) If t_1, t_2 are n-ary terms of type τ , then the equation $t_1 \approx t_2$ is an n-ary quantifier freeformula of type $(\tau,\acute{\tau})$.

(ii) If $j \in J$and $t_1, \ldots, t_{n_j}$ are n-ary terms of type τ and γ_j is an n_j-ary relation symbol, then $\gamma_j(t_1, \ldots, t_{n_j})$ is an n-ary quantifier free formula of type $(\tau,\acute{\tau})$.

(iii)If F is an n-ary quantifier free formula of type $(\tau,\acute{\tau})$, then $\neg F$ is an n-ary quantifier free formula of type $(\tau,\acute{\tau})$.

(iv) If F_1 and F_2 are n-ary quantifier free formulas of type $(\tau,\acute{\tau})$, then $F_1 \vee F_2$ is an n-ary quantifier free formula of type $(\tau,\acute{\tau})$.

Let $\mathcal{F}_{(\tau,\acute{\tau})}(W_\tau(X_n))$ be the set of all n-ary quantifier free formulas of type $(\tau,\acute{\tau})$. In 2012, M. Couceiro and E. Lehtonen introduced the concept of a linear term, i.e., a term which each variable occurs only once.

Definition 1.3. (Couceiro & Lehtonen, 2012) An n-ary linear term of type τ is defined inductively as follows:

(i) Every $x_i \in X_n$ is an n-ary linear term of type τ.

(ii) If $t_1, \ldots, t_{n_i}$ are n-ary linear terms of type τ with $var(t_l) \cap var(t_k) = \emptyset$ for all $1 \le l < k \le n_i$ (where $var(t)$ is the set of all variables occurring in the the term t) and f_i is an n_i-ary operation symbol, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary linear term of type τ.

Let $W_\tau^{lin}(X_n)$ be the set of all$n$-ary linear terms of type τ.

In this paper, we consider an algebraic system of type $((n),(2))$, i.e., we have only one n-ary operation symbol and one binary relation symbol. We define the new definition of linear formulas of type $((n),(2))$ and give the concept of superposition of linear terms and superposition of linear formulas. This leads to introduce the concept of linear hypersubstitutions for algebraic systems of type $((n),(2))$. We show that the set of all linear hypersubstitutions for algebraic systems of type $((n),(2))$ together with a binary operation $\circ_r$ and an identity element forms a monoid. Furthermore, the characterizations of idempotent and regular elements are investigated.

2. Linear Terms of Type (n) and Linear Formulas of Type $((n),(2))$

Let $var(t)$ be the set of all variables occurring in the term t and let $var(F)$ be the set of all variables occurring in the formula F.

In this section, we first defined the definition of a linear term and a quantifier free linear formula of type $((n),(2))$ as follows:

Definition 2.1. Let $m, n \in \mathbb{N}^+$with $m \ge n$. An m-ary linear term of type (n) is defined in the following inductive way:

(i) Every $x_i \in X_m$is an m-ary linear term of type (n).

(ii) If $t_1, \ldots, t_n$ are m-ary linear terms of type (n) with $var(t_l) \cap var(t_k) = \emptyset$ for all $1 \le l < k \le n$ and f is an n-ary operation symbol, then $f(t_1, \ldots, t_n)$ is an m-ary linear term of type (n).

Let $W_{(n)}^{lin}(X_m)$ be the set of all m-ary linear terms of type (n).

Example 2.2. Let $(n) = (2)$ be the type with a binary operation symbol f and $X_2 = \{x_1, x_2\}$. Then $x_1, x_2, f(x_1, x_2), f(x_2, x_1)$ are examples of binary linear terms of type (2).

Definition 2.3. Let $m, n \in \mathbb{N}^+$ with $m \geq n$. An m-ary quantifier free linear formula of type $((n),(2))$ (for simply, linear formula) is defined in the following steps:

(i) If t_1, t_2 are m-ary terms of type (n) and $var(t_1) \cap var(t_2) = \emptyset$ then the equation $t_1 \approx t_2$ is an m-ary quantifier free formula of type $((n),(2))$.

(ii) If t_1, t_2 are m-ary terms of type (n) with $var(t_1) \cap var(t_2) = \emptyset$and γ is a binary relation symbol, then $\gamma(t_1, t_2)$ is an m-ary quantifier free formula of type $((n),(2))$.

(iii)If F is an m-ary quantifier free formula of type $((n),(2))$, then $\neg F$ is an m-ary quantifier free formula of type $((n),(2))$.

(iv) If F_1 and F_2 are m-ary quantifier free formulas of type $((n),(2))$, then $F_1 \vee F_2$ is an m-ary quantifier free formula of type $((n),(2))$.

Let $\mathcal{F}^{lin}_{((n),(2))}(W_{(n)}(X_m))$ be the set of all m-ary quantifier free linear formulas of type $((n),(2))$.

Remark 2.4. The linear formulas defined by (i) and (ii) are called atomic linear formulas.

Example 2.5. Let $((2),(2))$ be a type, i.e., we have one binary operation symbol $f$and one binary relation symbol γ and let $X_2 = \{x_1, x_2\}$. Then the binary atomic linear formulas of type $((2),(2))$ are $x_1 \approx x_2$, $x_2 \approx x_1$, $\gamma(x_1, x_2)$, $\gamma(x_2, x_1)$. Moreover, we obtained all other linear formulas of type $((2),(2))$ from binary atomic linear formulas of type $((2),(2))$ by using the connectives $\neg$ and $\vee$.

Next, we give the concepts of the superposition of linear terms and linear formulas for algebraic systems of type $((n),(2))$. For convenient, we let $S_n$be the set of all permutations of $\{1, \ldots, n\}$.

3. Superposition of Linear terms and Linear Formulas

Definition 3.1. Let $m, n \in \mathbb{N}^+$with $m \geq n$, $t \in W^{lin}_{(n)}(X_n)$ and $t_1, \ldots, t_n \in W^{lin}_{(n)}(X_m)$ with $var(t_l) \cap var(t_k) = \emptyset$ for all $1 \leq l < k \leq n$. We define the concept of a superposition of linear terms

$$S^{lin}{}^{n}_{m}: W^{lin}_{(n)}(X_n) \times (W^{lin}_{(n)}(X_m))^n \multimap\!\to W^{lin}_{(n)}(X_m)$$

as follows:

(i) If $t = x_i\,; 1 \leq i \leq n$, then $S^{lin}{}^{n}_{m}(x_i, t_1, \ldots, t_n) := t_i$.

(ii) If $t = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$ where $\pi \in S_n$, then

$$S^{lin}{}^{n}_{m}(f(x_{\pi(1)}, \ldots, x_{\pi(n)}), t_1, \ldots, t_n) := f(S^{lin}{}^{n}_{m}(x_{\pi(1)}, t_1, \ldots, t_n), \ldots, S^{lin}{}^{n}_{m}(x_{\pi(n)}, t_1, \ldots, t_n)).$$

Now, we can extend the concept of this superposition to quantifier free linear formulas by substituting variables occurring in a quantifier free linear formula by a linear term, and obtain a new quantifier free linear formula. We explain this by the following operations $R^{lin}{}^{n}_{m}$when $m, n \geq 1$.

Definition 3.2. Let $m, n \in \mathbb{N}^+$ with $m \geq n$ and $t_1, \dots, t_n \in W_{(n)}^{lin}(X_m)$ with $var(t_l) \cap var(t_k) = \emptyset$ for all $1 \leq l < k \leq n$. The operation

$$R^{lin}{}^{n}_{m}: W_{(n)}^{lin}(X_n) \cup \mathcal{F}_{((n),(2))}^{lin}(W_{(n)}(X_n)) \times \left(W_{(n)}^{lin}(X_m)\right)^n \multimap\!\rightarrow W_{(n)}^{lin}(X_m) \cup \mathcal{F}_{((n),(2))}^{lin}(W_{(n)}(X_m))$$

is defined by the following inductive steps:

(i) If $t \in W_{(n)}^{lin}(X_n)$, then $R^{lin}{}^{n}_{m}(t, t_1, \dots, t_n) \coloneqq S^{lin}{}^{n}_{m}(t, t_1, \dots, t_n)$.

(ii) If $F \in \mathcal{F}_{((n),(2))}^{lin}\left(W_{(n)}(X_n)\right)$ and F has the form $x_{\pi(i)} \approx x_{\pi(j)}$ where $\pi \in S_n$, then $R^{lin}{}^{n}_{m}\left(x_{\pi(i)} \approx x_{\pi(j)}, t_1, \dots, t_n\right) \coloneqq S^{lin}{}^{n}_{m}\left(x_{\pi(i)}, t_1, \dots, t_n\right) \approx S^{lin}{}^{n}_{m}\left(x_{\pi(j)}, t_1, \dots, t_n\right)$.

(iii) If $F \in \mathcal{F}_{((n),(2))}^{lin}\left(W_{(n)}(X_n)\right)$ and F has the form $\gamma(x_{\pi(i)}, x_{\pi(j)})$ where $\pi \in S_n$, then $R^{lin}{}^{n}_{m}\left(\gamma(x_{\pi(i)}, x_{\pi(j)}), t_1, \dots, t_n\right) \coloneqq \gamma(S^{lin}{}^{n}_{m}\left(x_{\pi(i)}, t_1, \dots, t_n\right), S^{lin}{}^{n}_{m}\left(x_{\pi(j)}, t_1, \dots, t_n\right))$.

(iv) If $F \in \mathcal{F}_{((n),(2))}^{lin}\left(W_{(n)}(X_n)\right)$ and supposed that $R^{lin}{}^{n}_{m}(F, t_1, \dots, t_n)$ is already defined, then $R^{lin}{}^{n}_{m}(\neg F, t_1, \dots, t_n) \coloneqq \neg\left(R^{lin}{}^{n}_{m}(F, t_1, \dots, t_n)\right)$.

(v) If $F \in \mathcal{F}_{((n),(2))}^{lin}\left(W_{(n)}(X_n)\right)$ and F has the form $F_1 \vee F_2$ and supposed that $R^{lin}{}^{n}_{m}(F_l, t_1, \dots, t_n)$ is already defined for all $l = 1{,}2$, then $R^{lin}{}^{n}_{m}(F_1 \vee F_2, t_1, \dots, t_n) \coloneqq R^{lin}{}^{n}_{m}(F_1, t_1, \dots, t_n) \vee R^{lin}{}^{n}_{m}(F_2, t_1, \dots, t_n)$.

The next theorem is to show some properties of superposition of linear terms and superposition of linear formulas. We will use this theorem to prove the endomorphism properties of the extension of linear hypersubstitutions, identity linear hypersubstitution and some characterizations of special elements in the next section.

Theorem 3.3. Let $m, n, p \in \mathbb{N}^+$ with $m \geq n \geq p$. If $\beta \in W_{(n)}^{lin}(X_n) \cup \mathcal{F}_{((n),(2))}^{lin}\left(W_{(n)}(X_n)\right)$, then the operation $R^{lin}{}^{n}_{m}$ satisfies the following properties:

(FC1) $R^{lin}{}^{n}_{m}\left(R^{lin}{}^{p}_{n}(\beta, t_1, \dots, t_p), s_1, \dots, s_n\right) = R^{lin}{}^{p}_{m}\left(\beta, S^{lin}{}^{n}_{m}(t_1, s_1, \dots, s_n), \dots, S^{lin}{}^{n}_{m}(t_n, s_1, \dots, s_n)\right)$ where $t_1, \dots, t_p \in W_{(n)}(X_p)$, $var(t_l) \cap var(t_k) = \emptyset$ for all $1 \leq l < k \leq p$ and $s_1, \dots, s_n \in W_{(n)}(X_n)$, $var(s_l) \cap var(s_k) = \emptyset$ for all $1 \leq l < k \leq n$.

(FC2) $R^{lin}{}^{n}_{n}(\beta, x_1, \dots, x_n) = \beta$.

4. Monoid of Linear Hypersubstitutions for Algebraic Systems of Type $((n),(2))$

In this section, we would like to form the new structure of so-called "Monoid of Linear Hypersubstitutions for Algebraic Systems of Type $((n),(2))$". The way to approach this, we first define the based set.

Definition 4.1. Let $n \in \mathbb{N}^+$. A linear hypersubstitution for algebraic systems of type $((n),(2))$ is a mapping

$$\sigma: \{f\} \cup \{\gamma\} \rightarrow W_{(n)}^{lin}(X_n) \cup \mathcal{F}_{((n),(2))}^{lin}\left(W_{(n)}(X_2)\right)$$

which maps an n-ary operation symbol f to an n-ary linear term of type (n) and maps a binary relation symbol γ to a binary quantifier free linear formula of type $((n),(2))$. We denote the set of all linear hypersubstitutions for algebraic systems of type $((n),(2))$ by $Hyp^{lin}((n),(2))$.

From now on, every element in $Hyp^{lin}((n),(2))$ will be denoted by $\sigma_{t,F}$, that means $\sigma_{t,F}(f) = t$ and $\sigma_{t,F}(\gamma) = F$.

To define a binary operation on $Hyp^{lin}((n),(2))$, we extend a linear hypersubstitution for algebraic systems σ to a mapping $\hat{\sigma}$ defined by the following definition.

Definition 4.2. Let $\sigma_{t,F} \in Hyp^{lin}((n),(2))$, $\pi \in S_n$ and $\phi \in S_2$. Then we define a mapping

$$\hat{\sigma}_{t,F}: W_{(n)}^{lin}(X_n) \cup \mathcal{F}_{((n),(2))}^{lin}\left(W_{(n)}(X_2)\right) \to W_{(n)}^{lin}(X_n) \cup \mathcal{F}_{((n),(2))}^{lin}\left(W_{(n)}(X_2)\right)$$

inductively defined as follows:

(i) $\hat{\sigma}_{t,F}[x_i] := x_i$ for every $i = 1, \dots, n$.

(ii) $\hat{\sigma}_{t,F}[f(x_{\pi(1)}, \dots, x_{\pi(n)})] := S^{lin}{}^{n}_{n}(\sigma_{t,F}(f), \hat{\sigma}_{t,F}[x_{\pi(1)}], \dots, \hat{\sigma}_{t,F}[x_{\pi(n)}])$.

(iii) $\hat{\sigma}_{t,F}[x_{\phi(1)} \approx x_{\phi(2)}] := \hat{\sigma}_{t,F}[x_{\phi(1)}] \approx \hat{\sigma}_{t,F}[x_{\phi(2)}]$.

(iv) $\hat{\sigma}_{t,F}[\gamma(x_{\phi(1)}, x_{\phi(2)})] := R^{lin}{}^{2}_{2}(\sigma_{t,F}(\gamma), \hat{\sigma}_{t,F}[x_{\phi(1)}], \hat{\sigma}_{t,F}[x_{\phi(2)}])$.

(v) $\hat{\sigma}_{t,F}[\neg F] := \neg\hat{\sigma}_{t,F}[F]$ for $F \in \mathcal{F}_{((n),(2))}^{lin}\left(W_{(n)}(X_2)\right)$.

Now, we define a binary operation $\circ_r$ on $Hyp^{lin}((n),(2))$ as follows:

Definition 4.3. Let $\sigma_{t_1,F_1}, \sigma_{t_2,F_2} \in Hyp^{lin}((n),(2))$ and $\circ$ be the usual composition of mapping. Then we define a binary operation $\circ_r$ on $Hyp^{lin}((n),(2))$ by $\sigma_{t_1,F_1} \circ_r \sigma_{t_2,F_2} := \hat{\sigma}_{t_1,F_1} \circ \sigma_{t_2,F_2}$.

Next, we prove that a binary operation as we already defined in Definition 4.3 satisfies associative law. To get our result, we need some preparations as follows:

Lemma 4.4. For each $\sigma_{t,F} \in Hyp^{lin}((n),(2))$, $\pi \in S_n$ and $\phi \in S_2$. Then we have

(i) $\hat{\sigma}_{t,F}[S^{lin}{}^{n}_{n}(t, x_{\pi(1)}, \dots, x_{\pi(n)})] = S^{lin}{}^{n}_{n}(\hat{\sigma}_{t,F}[t], \hat{\sigma}_{t,F}[x_{\pi(1)}], \dots, \hat{\sigma}_{t,F}[x_{\pi(n)}])$.

(ii) $\hat{\sigma}_{t,F}[R^{lin}{}^{2}_{2}(\beta, x_{\phi(1)}, x_{\phi(2)})] = R^{lin}{}^{2}_{2}(\hat{\sigma}_{t,F}[\beta], \hat{\sigma}_{t,F}[x_{\phi(1)}], \hat{\sigma}_{t,F}[x_{\phi(2)}])$.

Proof. (i) Let $t \in W_{(n)}^{lin}(X_n)$. We give a proof by induction on the complexity of a linear term t. Obviously, if $t = x_i$ for all $1 \le i \le n$. If $t = f(x_{\pi(1)}, \dots, x_{\pi(n)})$ and for every $l = 1, \dots, n$ we assume that $\hat{\sigma}_{t,F}[S^{lin}{}^{n}_{n}(x_{\pi(l)}, x_{\pi(1)}, \dots, x_{\pi(n)})]$ $= S^{lin}{}^{n}_{n}(\hat{\sigma}_{t,F}[x_{\pi(l)}], \hat{\sigma}_{t,F}[x_{\pi(1)}], \dots, \hat{\sigma}_{t,F}[x_{\pi(n)}])$, then by Theorm 3.3 we get $\hat{\sigma}_{t,F}[S^{lin}{}^{n}_{n}(f(x_{\pi(1)}, \dots, x_{\pi(n)}), x_{\pi(1)}, \dots, x_{\pi(n)})]$

$=\hat{\sigma}_{t,F}[f(S^{lin}{}^{n}_{n}(x_{\pi(1)}, x_{\pi(1)}, \dots, x_{\pi(n)}), \dots, f(S^{lin}{}^{n}_{n}(x_{\pi(n)}, x_{\pi(1)}, \dots, x_{\pi(n)}))]$

$=S^{lin}{}^{n}_{n}(\sigma_{t,F}(f), \hat{\sigma}_{t,F}[S^{lin}{}^{n}_{n}(x_{\pi(1)}, x_{\pi(1)}, \dots, x_{\pi(n)})], \dots, \hat{\sigma}_{t,F}[S^{lin}{}^{n}_{n}(x_{\pi(n)}, x_{\pi(1)}, \dots, x_{\pi(n)})])$

$=S^{lin}{}^{n}_{n}(\sigma_{t,F}(f), S^{lin}{}^{n}_{n}(\hat{\sigma}_{t,F}[x_{\pi(1)}], \hat{\sigma}_{t,F}[x_{\pi(1)}], \dots, \hat{\sigma}_{t,F}[x_{\pi(n)}]), \dots, S^{lin}{}^{n}_{n}(\hat{\sigma}_{t,F}[x_{\pi(n)}], \hat{\sigma}_{t,F}[x_{\pi(1)}], \dots, \hat{\sigma}_{t,F}[x_{\pi(n)}])$.

$=S^{lin}{}^{n}_{n}(S^{lin}{}^{n}_{n}(\sigma_{t,F}(f), \hat{\sigma}_{t,F}[x_{\pi(1)}], \dots, \hat{\sigma}_{t,F}[x_{\pi(n)}]), \hat{\sigma}_{t,F}[x_{\pi(1)}], \dots, \hat{\sigma}_{t,F}[x_{\pi(n)}])$

$=S^{lin}{}^{n}_{n}(\hat{\sigma}_{t,F}[f(x_{\pi(1)}, \dots, x_{\pi(n)})], \hat{\sigma}_{t,F}[x_{\pi(1)}], \dots, \hat{\sigma}_{t,F}[x_{\pi(n)}])$.

(ii) Let $\beta \in \mathcal{F}_{((n),(2))}^{lin}\left(W_{(n)}(X_2)\right)$. We give a proof by the following steps.

If β has the form $x_{\phi(1)} \approx x_{\phi(2)}$, then we have $\hat{\sigma}_{t,F}[R^{lin}{}^{2}_{2}(x_{\phi(1)} \approx x_{\phi(2)}, x_{\phi(1)}, x_{\phi(2)})]$

$=\hat{\sigma}_{t,F}[S^{lin}{}^{2}_{2}(x_{\phi(1)}, x_{\phi(1)}, x_{\phi(2)}) \approx S^{lin}{}^{2}_{2}(x_{\phi(2)}, x_{\phi(1)}, x_{\phi(2)}))]$

$=S^{lin}{}^{2}_{2}(\hat{\sigma}_{t,F}[x_{\phi(1)}], \hat{\sigma}_{t,F}[x_{\phi(1)}], \hat{\sigma}_{t,F}[x_{\phi(2)}]) \approx S^{lin}{}^{2}_{2}(\hat{\sigma}_{t,F}[x_{\phi(2)}], \hat{\sigma}_{t,F}[x_{\phi(1)}], \hat{\sigma}_{t,F}[x_{\phi(2)}])$

$=R^{lin}{}^{2}_{2}(\hat{\sigma}_{t,F}[x_{\phi(1)}] \approx \hat{\sigma}_{t,F}[x_{\phi(2)}], \hat{\sigma}_{t,F}[x_{\phi(1)}], \hat{\sigma}_{t,F}[x_{\phi(2)}])$

$=R^{lin}{}^{2}_{2}(\hat{\sigma}_{t,F}[x_{\phi(1)} \approx x_{\phi(2)}], \hat{\sigma}_{t,F}[x_{\phi(1)}], \hat{\sigma}_{t,F}[x_{\phi(2)}])$.

If β has the form $\gamma(x_{\phi(1)}, x_{\phi(2)})$, then by Theorem 3.3 we have $\hat{\sigma}_{t,F}\left[R^{lin}{}^2_2\left(\gamma(x_{\phi(1)}, x_{\phi(2)}), x_{\phi(1)}, x_{\phi(2)}\right)\right]$

$=\hat{\sigma}_{t,F}\left[\gamma(S^{lin}{}^2_2\left(x_{\phi(1)}, x_{\phi(1)}, x_{\phi(2)}\right), \gamma(S^{lin}{}^2_2\left(x_{\phi(2)}, x_{\phi(1)}, x_{\phi(2)}\right))\right]$

$=R^{lin}{}^2_2(\sigma_{t,F}(\gamma), \hat{\sigma}_{t,F}\left[S^{lin}{}^2_2\left(x_{\phi(1)}, x_{\phi(1)}, x_{\phi(2)}\right)\right], \hat{\sigma}_{t,F}\left[S^{lin}{}^2_2\left(x_{\phi(2)}, x_{\phi(1)}, x_{\phi(2)}\right)\right])$

$=R^{lin}{}^2_2(\sigma_{t,F}(\gamma), S^{lin}{}^2_2(\hat{\sigma}_{t,F}[x_{\phi(1)}], \hat{\sigma}_{t,F}[x_{\phi(1)}], \hat{\sigma}_{t,F}[x_{\phi(2)}]),\ S^{lin}{}^2_2(\hat{\sigma}_{t,F}[x_{\phi(2)}], \hat{\sigma}_{t,F}[x_{\phi(1)}], \hat{\sigma}_{t,F}[x_{\phi(2)}]).$

$=R^{lin}{}^2_2(R^{lin}{}^2_2\left(\sigma_{t,F}(\gamma), \hat{\sigma}_{t,F}[x_{\phi(1)}], \hat{\sigma}_{t,F}[x_{\phi(2)}]\right), \hat{\sigma}_{t,F}[x_{\phi(1)}], \hat{\sigma}_{t,F}[x_{\phi(2)}]\ )$

$=R^{lin}{}^2_2(\hat{\sigma}_{t,F}[\gamma(x_{\phi(1)}, x_{\phi(2)})], \hat{\sigma}_{t,F}[x_{\phi(1)}], \hat{\sigma}_{t,F}[x_{\phi(2)}]).$

If β has the form $\neg F$and assume that $\hat{\sigma}_{t,F}\left[R^{lin}{}^2_2\left(F, x_{\phi(1)}, x_{\phi(2)}\right)\right] = R^{lin}{}^2_2(\hat{\sigma}_{t,F}[F], \hat{\sigma}_{t,F}[x_{\phi(1)}], \hat{\sigma}_{t,F}[x_{\phi(2)}])$, then we get $\hat{\sigma}_{t,F}\left[R^{lin}{}^2_2\left(\neg F, x_{\phi(1)}, x_{\phi(2)}\right)\right] = \hat{\sigma}_{t,F}\left[\neg(R^{lin}{}^2_2\left(F, x_{\phi(1)}, x_{\phi(2)}\right))\right] = \neg(\hat{\sigma}_{t,F}\left[R^{lin}{}^2_2\left(F, x_{\phi(1)}, x_{\phi(2)}\right)\right]) =$
$\neg(R^{lin}{}^2_2\left(\hat{\sigma}_{t,F}[F], \hat{\sigma}_{t,F}[x_{\phi(1)}], \hat{\sigma}_{t,F}[x_{\phi(2)}]\right) = R^{lin}{}^2_2(\neg\hat{\sigma}_{t,F}[F], \hat{\sigma}_{t,F}[x_{\phi(1)}], \hat{\sigma}_{t,F}[x_{\phi(2)}])$
$= R^{lin}{}^2_2(\hat{\sigma}_{t,F}[\neg F], \hat{\sigma}_{t,F}[x_{\phi(1)}], \hat{\sigma}_{t,F}[x_{\phi(2)}]).$

As a result of Lemma 4.4, we have the following lemma.

Lemma 4.5. Let $\sigma_{t_1,F_1}, \sigma_{t_2,F_2} \in Hyp^{lin}((n),(2))$. Then we have

$$(\sigma_{t_1,F_1} \circ_r \sigma_{t_2,F_2})^\wedge = \hat{\sigma}_{t_1,F_1} \circ \hat{\sigma}_{t_2,F_2}.$$

Proof. Let $t \in W^{lin}_{(n)}(X_n)$, we give a proof by induction on the complexity of a linear term t. If $t = x_i\,; 1 \le i \le n$, then $(\sigma_{t_1,F_1} \circ_r \sigma_{t_2,F_2})^\wedge[x_i] = x_i = \hat{\sigma}_{t_1,F_1}[x_i] = \hat{\sigma}_{t_1,F_1}[\hat{\sigma}_{t_2,F_2}[x_i]] = (\hat{\sigma}_{t_1,F_1} \circ \hat{\sigma}_{t_2,F_2})[x_i]$. If $t = f(x_{\pi(1)}, \dots, x_{\pi(n)})$, then by Lemma 4.4 we have that $(\sigma_{t_1,F_1} \circ_r \sigma_{t_2,F_2})^\wedge[f(x_{\pi(1)}, \dots, x_{\pi(n)})]$

$=S^{lin}{}^n_n((\sigma_{t_1,F_1} \circ_r \sigma_{t_2,F_2})(f), (\sigma_{t_1,F_1} \circ_r \sigma_{t_2,F_2})^\wedge[x_{\pi(1)}], \dots, (\sigma_{t_1,F_1} \circ_r \sigma_{t_2,F_2})^\wedge[x_{\pi(n)}])$

$=S^{lin}{}^n_n(\hat{\sigma}_{t_1,F_1}[\sigma_{t_2,F_2}(f)], \hat{\sigma}_{t_1,F_1}[\hat{\sigma}_{t_2,F_2}[x_{\pi(1)}]], \dots, \hat{\sigma}_{t_1,F_1}[\hat{\sigma}_{t_2,F_2}[x_{\pi(n)}]])$

$=\hat{\sigma}_{t_1,F_1}[S^{lin}{}^n_n(\sigma_{t_2,F_2}(f), \hat{\sigma}_{t_2,F_2}[x_{\pi(1)}], \dots, \hat{\sigma}_{t_2,F_2}[x_{\pi(n)}])]$

$=\hat{\sigma}_{t_1,F_1}[\hat{\sigma}_{t_2,F_2}[f(x_{\pi(1)}, \dots, x_{\pi(n)})]]$

$= (\hat{\sigma}_{t_1,F_1} \circ \hat{\sigma}_{t_2,F_2})[f(x_{\pi(1)}, \dots, x_{\pi(n)})].$

Let $\beta \in \mathcal{F}^{lin}_{((n),(2))}\left(W_{(n)}(X_2)\right)$. We give a proof by the following steps.

If β has the form $x_{\phi(1)} \approx x_{\phi(2)}$, then we have $(\sigma_{t_1,F_1} \circ_r \sigma_{t_2,F_2})^\wedge[x_{\phi(1)} \approx x_{\phi(2)}]$

$= (\sigma_{t_1,F_1} \circ_r \sigma_{t_2,F_2})^\wedge[x_{\phi(1)}] \approx (\sigma_{t_1,F_1} \circ_r \sigma_{t_2,F_2})^\wedge[x_{\phi(2)}]$

$= (\hat{\sigma}_{t_1,F_1} \circ \hat{\sigma}_{t_2,F_2})[x_{\phi(1)}] \approx (\hat{\sigma}_{t_1,F_1} \circ \hat{\sigma}_{t_2,F_2})[x_{\phi(2)}]$

$= \hat{\sigma}_{t_1,F_1}[\hat{\sigma}_{t_2,F_2}[x_{\phi(1)}]] \approx \hat{\sigma}_{t_1,F_1}[\hat{\sigma}_{t_2,F_2}[x_{\phi(2)}]]$

$= \hat{\sigma}_{t_1,F_1}[x_{\phi(1)}] \approx \hat{\sigma}_{t_1,F_1}[x_{\phi(2)}]$

$= x_{\phi(1)} \approx x_{\phi(2)}$

$= (\hat{\sigma}_{t_1,F_1} \circ \hat{\sigma}_{t_2,F_2})[x_{\phi(1)} \approx x_{\phi(2)}].$

If β has the form $\gamma(x_{\phi(1)}, x_{\phi(2)})$, then by Lemma 4.4 we have that $(\sigma_{t_1,F_1} \circ_r \sigma_{t_2,F_2})^\wedge[\gamma(x_{\phi(1)}, x_{\phi(2)})]$

$=R^{lin}{}^2_2((\sigma_{t_1,F_1} \circ_r \sigma_{t_2,F_2})(\gamma), (\sigma_{t_1,F_1} \circ_r \sigma_{t_2,F_2})^\wedge[x_{\phi(1)}], (\sigma_{t_1,F_1} \circ_r \sigma_{t_2,F_2})^\wedge[x_{\phi(2)}])$

$=R^{lin}{}^2_2(\hat{\sigma}_{t_1,F_1}[\sigma_{t_2,F_2}(\gamma)], \hat{\sigma}_{t_1,F_1}[\hat{\sigma}_{t_2,F_2}[x_{\phi(1)}]], \hat{\sigma}_{t_1,F_1}[\hat{\sigma}_{t_2,F_2}[x_{\phi(2)}]])$

$=\hat{\sigma}_{t_1,F_1}[R^{lin}{}^2_2(\sigma_{t_2,F_2}(\gamma), \hat{\sigma}_{t_2,F_2}[x_{\phi(1)}], \hat{\sigma}_{t_2,F_2}[x_{\phi(2)}])]$

$=\hat{\sigma}_{t_1,F_1}[\hat{\sigma}_{t_2,F_2}[\gamma(x_{\phi(1)}, x_{\phi(2)})]]$

$= (\hat{\sigma}_{t_1,F_1} \circ \hat{\sigma}_{t_2,F_2})[\gamma(x_{\phi(1)}, x_{\phi(2)})]$.

If β has the form$\neg F$ and assume that $(\sigma_{t_1,F_1} \circ_r \sigma_{t_2,F_2})^\wedge[F] = (\hat{\sigma}_{t_1,F_1} \circ \hat{\sigma}_{t_2,F_2})[F]$, then we obtain that

$(\sigma_{t_1,F_1} \circ_r \sigma_{t_2,F_2})^\wedge[\neg F] = \neg(\sigma_{t_1,F_1} \circ_r \sigma_{t_2,F_2})^\wedge[F] = \neg(\hat{\sigma}_{t_1,F_1} \circ \hat{\sigma}_{t_2,F_2})[F] = \neg\hat{\sigma}_{t_1,F_1}\left[\hat{\sigma}_{t_2,F_2}[F]\right] = \hat{\sigma}_{t_1,F_1}\left[\neg\hat{\sigma}_{t_2,F_2}[F]\right] =$
$\hat{\sigma}_{t_1,F_1}\left[\hat{\sigma}_{t_2,F_2}[\neg F]\right] = (\hat{\sigma}_{t_1,F_1} \circ \hat{\sigma}_{t_2,F_2})[\neg F]$.

It follows from Lemma 4.5 that the binary operation $\circ_r$ satisfies associative law. We prove this fact in the next lemma.

Lemma 4.6. For any $\sigma_{t_1,F_1}, \sigma_{t_2,F_2}, \sigma_{t_3,F_3} \in Hyp^{lin}((n),(2))$, we hav

$$(\sigma_{t_1,F_1} \circ_r \sigma_{t_2,F_2}) \circ_r \sigma_{t_3,F_3} = \sigma_{t_1,F_1} \circ_r (\sigma_{t_2,F_2} \circ_r \sigma_{t_3,F_3}).$$

Proof. By using Lemma 4.5 and the fact that $\circ$satisfies associative law, it can be shown that $\circ_r$ satisfies associative law. In fact, we have $(\sigma_{t_1,F_1} \circ_r \sigma_{t_2,F_2}) \circ_r \sigma_{t_3,F_3} = (\sigma_{t_1,F_1} \circ_r \sigma_{t_2,F_2})^\wedge \circ \sigma_{t_3,F_3} = (\hat{\sigma}_{t_1,F_1} \circ \hat{\sigma}_{t_2,F_2}) \circ \sigma_{t_3,F_3} = \hat{\sigma}_{t_1,F_1} \circ (\hat{\sigma}_{t_2,F_2} \circ \sigma_{t_3,F_3}) = \hat{\sigma}_{t_1,F_1} \circ (\sigma_{t_2,F_2} \circ_r \sigma_{t_3,F_3}) = \sigma_{t_1,F_1} \circ_r (\sigma_{t_2,F_2} \circ_r \sigma_{t_3,F_3})$.

Let σ_{id} be a linear hypersubstitution for algebraic systems which maps the operation symbol $f$to the linear term $f(x_1, \ldots, x_n)$and maps the relation symbol γ to the linear formula $\gamma(x_1, x_2)$, i.e. $\sigma_{id}(f) = f(x_1, \ldots, x_n)$ and $\sigma_{id}(\gamma) = \gamma(x_1, x_2)$.

A linear hypersubstitution σ_{id} is claimed to be an identity, which we will prove this fact in the next lemma.

Lemma 4.7. For any linear term $t \in W_{(n)}^{lin}(X_n)$ and linear formula

$\beta \in \mathcal{F}_{((n),(2))}^{lin}\left(W_{(n)}(X_2)\right)$, we have $\hat{\sigma}_{id}[t] = t$ and $\hat{\sigma}_{id}[\beta] = \beta$.

Proof. Let $t \in W_{(n)}^{lin}(X_n)$, we give a proof by induction on the complexity of a linear term t. If$t = x_i$with$i = 1, \ldots, n$, then $\hat{\sigma}_{id}[x_i] = x_i$. If $t = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$ where $\pi \in S_n$, then we get $\hat{\sigma}_{id}[f(x_{\pi(1)}, \ldots, x_{\pi(n)})] =$
$S^{lin}{}_n^n(\sigma_{t,F}(f), \hat{\sigma}_{t,F}[x_{\pi(1)}], \ldots, \hat{\sigma}_{t,F}[x_{\pi(n)}]) = S^{lin}{}_n^n(f(x_1, \ldots, x_n), x_{\pi(1)}, \ldots, x_{\pi(n)}) = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$. Next, let $\beta \in \mathcal{F}_{((n),(2))}^{lin}\left(W_{(n)}(X_2)\right), \phi \in S_2$, we give a proof by the following steps.

If β has the form $x_{\phi(1)} \approx x_{\phi(2)}$, then we have $\hat{\sigma}_{id}[x_{\phi(1)} \approx x_{\phi(2)}] = \hat{\sigma}_{id}[x_{\phi(1)}] \approx \hat{\sigma}_{id}[x_{\phi(1)}] = x_{\phi(1)} \approx x_{\phi(2)}$. If β has the form$\gamma(x_{\phi(1)}, x_{\phi(2)})$, then $\hat{\sigma}_{id}[\gamma(x_{\phi(1)}, x_{\phi(2)})] = R^{lin}{}_2^2(\sigma_{t,F}(\gamma), \hat{\sigma}_{t,F}[x_{\phi(1)}], \hat{\sigma}_{t,F}[x_{\phi(2)}]) = R^{lin}{}_2^2(\gamma(x_1, x_2), x_{\phi(1)}, x_{\phi(2)}) = \gamma(x_{\phi(1)}, x_{\phi(2)})$. If β has the form $\neg F$ and assume that $\hat{\sigma}_{id}[F] = F$, then $\hat{\sigma}_{id}[\neg F] = \neg\hat{\sigma}_{id}[F] = \neg F$.

Lemma 4.8. Let $\sigma_{id} \in Hyp^{lin}((n),(2))$.Then we have $\sigma_{id}$is an identity element with respect to $\circ_r$.

Proof. First, we prove that σ_{id} is a left identity element by using Lemma 4.7. Let $\sigma_{t,F} \in Hyp^{lin}((n),(2))$. Then we have$(\sigma_{id} \circ_r \sigma_{t,F})(f) = (\hat{\sigma}_{id} \circ \sigma_{t,F})(f) = \hat{\sigma}_{id}[\sigma_{t,F}(f)] = \sigma_{t,F}(f)$ and $(\sigma_{id} \circ_r \sigma_{t,F})(\gamma) = (\hat{\sigma}_{id} \circ \sigma_{t,F})(\gamma) = \hat{\sigma}_{id}[\sigma_{t,F}(\gamma)] = \sigma_{t,F}(\gamma)$. Now, we show that σ_{id} is a right identity element. Let $\sigma_{t,F} \in Hyp^{lin}((n),(2))$. By Theorem 3.3, we obtain that $(\sigma_{t,F} \circ_r \sigma_{id})(f) = (\hat{\sigma}_{t,F} \circ \sigma_{id})(f) = \hat{\sigma}_{t,F}[\sigma_{id}(f)] = \hat{\sigma}_{t,F}[f(x_1, \ldots, x_n)] = S^{lin}{}_n^n(\sigma_{t,F}(f), \hat{\sigma}_{t,F}[x_1], \ldots, \hat{\sigma}_{t,F}[x_1]) =$
$S^{lin}{}_n^n(\sigma_{t,F}(f), x_1, \ldots, x_n) = \sigma_{t,F}(f)$ and $(\sigma_{t,F} \circ_r \sigma_{id})(\gamma) = (\hat{\sigma}_{t,F} \circ \sigma_{id})(\gamma) = \hat{\sigma}_{t,F}[\sigma_{id}(\gamma)] = \hat{\sigma}_{t,F}[\gamma(x_1, x_2)]$
$= R^{lin}{}_2^2(\sigma_{t,F}(\gamma), \hat{\sigma}_{t,F}[x_1], \hat{\sigma}_{t,F}[x_2]) = R^{lin}{}_2^2(\sigma_{t,F}(\gamma), x_1, x_2) = \sigma_{t,F}(\gamma)$. This implies that $\sigma_{id} \circ_r \sigma_{t,F} = \sigma_{t,F} = \sigma_{t,F} \circ_r \sigma_{id}$.
Therefore, σ_{id} is an identity element.

Theorem 4.9. $\mathcal{H}yp^{lin}\big((n),(2)\big) := (Hyp^{lin}\big((n),(2)\big), \circ_r, \sigma_{id})$ is a monoid.

Proof. From Lemma 4.6 and 4.8, the conclusion holds.

5. Idempotent and Regular Elements in $\mathcal{H}yp^{lin}\big((n),(2)\big)$

In this section we study some semigroup properties of $\mathcal{H}yp^{lin}\big((n),(2)\big)$, especially we characterize idempotency and regularity of $\sigma_{t,F} \in Hyp^{lin}\big((n),(2)\big)$. We first introduce some notations and definitions of idempotent and regular elements in $Hyp^{lin}\big((n),(2)\big)$ with respect to $\circ_r$.

For any $\sigma_{t,F} \in Hyp^{lin}\big((n),(2)\big), \pi \in S_n, \phi \in S_2$ we denote :

$B_1 := \{\sigma_{t,F} \mid t = x_i \in X_n, F = x_{\phi(1)} \approx x_{\phi(2)}\}$,

$B_2 := \{\sigma_{t,F} \mid t = x_i \in X_n, F = \gamma(x_{\phi(1)}, x_{\phi(2)})\}$,

$B_3 := \{\sigma_{t,F} \mid t = x_i \in X_n, F = \neg(x_{\phi(1)} \approx x_{\phi(2)})\}$,

$B_4 := \{\sigma_{t,F} \mid t = x_i \in X_n, F = \neg\gamma(x_{\phi(1)}, x_{\phi(2)})\}$,

$B_5 := \{\sigma_{t,F} \mid t = f(x_{\pi(1)}, \ldots, x_{\pi(n)}), F = x_{\phi(1)} \approx x_{\phi(2)}\}$,

$B_6 := \{\sigma_{t,F} \mid t = f(x_{\pi(1)}, \ldots, x_{\pi(n)}), F = \gamma(x_{\phi(1)}, x_{\phi(2)})\}$,

$B_7 := \{\sigma_{t,F} \mid t = f(x_{\pi(1)}, \ldots, x_{\pi(n)}), F = \neg(x_{\phi(1)} \approx x_{\phi(2)})\}$,

$B_8 := \{\sigma_{t,F} \mid t = f(x_{\pi(1)}, \ldots, x_{\pi(n)}), F = \neg\gamma(x_{\phi(1)}, x_{\phi(2)})\}$.

We note that $P = \{B_1, \ldots, B_8\}$ is a partition of $Hyp^{lin}\big((n),(2)\big)$.

The concepts of an idempotent element and a regular element are defined in $\mathcal{H}yp^{lin}\big((n),(2)\big)$. An element $\sigma_{t,F} \in Hyp^{lin}\big((n),(2)\big)$ is said to *be idempotent* if $\sigma_{t,F} \circ_r \sigma_{t,F} = \sigma_{t,F}$, that is, $(\sigma_{t,F} \circ_r \sigma_{t,F})(f) = \sigma_{t,F}(f)$and $(\sigma_{t,F} \circ_r \sigma_{t,F})(\gamma) = \sigma_{t,F}(\gamma)$. And $\sigma_{t,F} \in Hyp^{lin}\big((n),(2)\big)$ is called *regular*if there is an element $\sigma_{\acute{t},\acute{F}} \in Hyp^{lin}\big((n),(2)\big)$ such that $\sigma_{t,F} = \sigma_{t,F} \circ_r \sigma_{\acute{t},\acute{F}} \circ_r \sigma_{t,F}$. The semigroup $\mathcal{H}yp^{lin}\big((n),(2)\big)$ is called regular if every element in $Hyp^{lin}\big((n),(2)\big)$ is regular. Furthermore, we denote the set of all idempotent and regular in $\mathcal{H}yp^{lin}\big((n),(2)\big)$ by $E\left(Hyp^{lin}\big((n),(2)\big)\right)$ and $Reg\left(Hyp^{lin}\big((n),(2)\big)\right)$, respectively.

Lemma 5.1. (Burris, 1981) Suppose F is a formula insome $\mathcal{F}_{(\tau,\acute{\tau})}(W_\tau(X_n))$. Then the following pair of formula is equivalent: $\neg(\neg F) \equiv F$.

Lemma 5.2. Let $\sigma_{t,F} \in Hyp^{lin}\big((n),(2)\big)$. Then $\sigma_{t,F}$ is idempotent if and only if $\hat{\sigma}_{t,F}[t] = t$ and $\hat{\sigma}_{t,F}[F] = F$.

Proof. Assume that $\sigma_{t,F}$ is idempotent, i.e., $(\sigma_{t,F} \circ_r \sigma_{t,F})(f) = \sigma_{t,F}(f)$ and $(\sigma_{t,F} \circ_r \sigma_{t,F})(\gamma) = \sigma_{t,F}(\gamma)$. We now consider $\hat{\sigma}_{t,F}[t] = \hat{\sigma}_{t,F}[\sigma_{t,F}(f)] = (\hat{\sigma}_{t,F} \circ \sigma_{t,F})(f) = (\sigma_{t,F} \circ_r \sigma_{t,F})(f) = \sigma_{t,F}(f) = t$ and $\hat{\sigma}_{t,F}[F] = \hat{\sigma}_{t,F}[\sigma_{t,F}(\gamma)] = (\hat{\sigma}_{t,F} \circ \sigma_{t,F})(\gamma) = (\sigma_{t,F} \circ_r \sigma_{t,F})(\gamma) = \sigma_{t,F}(\gamma) = F$. Conversely, let $\hat{\sigma}_{t,F}[t] = t$ and $\hat{\sigma}_{t,F}[F] = F$. Then we have$(\sigma_{t,F} \circ_r \sigma_{t,F})(f) = (\hat{\sigma}_{t,F} \circ \sigma_{t,F})(f) = \hat{\sigma}_{t,F}[\sigma_{t,F}(f)] = \hat{\sigma}_{t,F}[t] = t = \sigma_{t,F}(f)$ and $(\sigma_{t,F} \circ_r \sigma_{t,F})(\gamma) = (\hat{\sigma}_{t,F} \circ \sigma_{t,F})(\gamma) = \hat{\sigma}_{t,F}[\sigma_{t,F}(\gamma)] = \hat{\sigma}_{t,F}[F] = F = \sigma_{t,F}(\gamma)$. This shows that $\sigma_{t,F}$is idempotent.

Proposition 5.3. σ_{id} is idempotent.

Proof. Since σ_{id} is an identity in $Hyp^{lin}((n),(2))$ and by Lemma 4.7, we obtain that $\hat{\sigma}_{id}[t] = t$ and $\hat{\sigma}_{id}[F] = F$. By Lemma 5.2, we have that σ_{id} is idempotent.

Theorem 5.4. Let$\sigma_{t,F} \in Hyp^{lin}((n),(2))$. Then the following statements hold.

(i) Every $\sigma_{t,F} \in B_1$ is idempotent.

(ii) Every $\sigma_{t,F} \in B_3$ is idempotent.

(iii) Every $\sigma_{t,F} \in B_4$ is not idempotent.

Proof. We first prove that $\sigma_{t,F} \in B_1$ is idempotent. To do this, let $\sigma_{t,F} \in B_1$ with $t = x_i$ and $F = x_{\phi(1)} \approx x_{\phi(2)}$. We consider $\hat{\sigma}_{t,F}[x_i] = x_i$ and $\hat{\sigma}_{t,F}[x_{\phi(1)} \approx x_{\phi(2)}] = \hat{\sigma}_{t,F}[x_{\phi(1)}] \approx \hat{\sigma}_{t,F}[x_{\phi(2)}] = x_{\phi(1)} \approx x_{\phi(2)}$. By Lemma 5.2, $\sigma_{t,F} \in B_1$ is idempotent. Next, let $\sigma_{t,F} \in B_3$ with $t = x_i$ and $F = \neg(x_{\phi(1)} \approx x_{\phi(2)})$. We consider $\hat{\sigma}_{t,F}[x_i] = x_i$ and $\hat{\sigma}_{t,F}[\neg(x_{\phi(1)} \approx x_{\phi(2)})] = \neg(\hat{\sigma}_{t,F}[x_{\phi(1)} \approx x_{\phi(2)}]) = \neg(x_{\phi(1)} \approx x_{\phi(2)})$ and then by Lemma 5.2, $\sigma_{t,F} \in B_3$ is idempotent. Lastly, let $\sigma_{t,F} \in B_4$ with $t = x_i$ and $F = \neg\gamma(x_{\phi(1)}, x_{\phi(2)})$. To show that it is not idempotent, we consider

$$
\begin{aligned}
\hat{\sigma}_{t,F}[\neg\gamma(x_{\phi(1)}, x_{\phi(2)})] &= \neg(\hat{\sigma}_{t,F}[\gamma(x_{\phi(1)}, x_{\phi(2)})]) \\
&= \neg\left(R^{lin}{}^2_2(\sigma_{t,F}(\gamma), \hat{\sigma}_{t,F}[x_{\phi(1)}], \hat{\sigma}_{t,F}[x_{\phi(2)}])\right) \\
&= \neg\left(R^{lin}{}^2_2(\neg\gamma(x_{\phi(1)}, x_{\phi(2)}), x_{\phi(1)}, x_{\phi(2)})\right) \\
&= \neg\left(\neg\left(R^{lin}{}^2_2(\gamma(x_{\phi(1)}, x_{\phi(2)}), x_{\phi(1)}, x_{\phi(2)})\right)\right) \\
&= \gamma\left(x_{\phi(\phi(1))}, x_{\phi(\phi(2))}\right) \\
&\neq \neg\gamma(x_{\phi(1)}, x_{\phi(2)}).
\end{aligned}
$$

Therefore, every $\sigma_{t,F} \in B_4$ is not idempotent.

The following example shows that there is an element in B_2 which is not idempotent.

Example 5.5. Let ((3), (2)) be a type, i.e., we have one ternary operation symbol and one binary relation symbol, say f and γ, respectively. If we consider $\sigma_{t,F} \in B_2$ with $t = x_2$ and $F = \gamma(x_2, x_1)$, then we obtain $\hat{\sigma}_{t,F}[\gamma(x_2, x_1)] = R^{lin}{}^2_2(\sigma_{t,F}(\gamma), x_2, x_1) = R^{lin}{}^2_2(\gamma(x_2, x_1), x_2, x_1) = \gamma(x_1, x_2) \neq \gamma(x_2, x_1)$. So, $\sigma_{t,F}$ in this form is not idempotent.

We have to find some necessary conditions for the element in B_2 which is idempotent element. The next theorem shows such condition.

Theorem 5.6. Let$\sigma_{t,F} \in B_2$. Then $\sigma_{t,F}$ is idempotent if and only if $\phi(j) = j$ for all $j = 1,2$.

Proof. Let $\sigma_{t,F} \in B_2$. Then we have $t = x_i$ and $F = \gamma(x_{\phi(1)}, x_{\phi(2)})$. Assume that $\phi(j) \neq j$ for some $j = 1,2$. We prove that $\sigma_{t,F}$ is not idempotent. To show this, we consider $\hat{\sigma}_{t,F}[\gamma(x_{\phi(1)}, x_{\phi(2)})] = \hat{\sigma}_{t,F}[\gamma(x_2, x_1)] = R^{lin}{}^2_2(\gamma(x_2, x_1), x_2, x_1) = \gamma(x_1, x_2) \neq \gamma(x_2, x_1)$ and then by Lemma 5.2, $\sigma_{t,F}$ is not idempotent. Conversely, assume that the condition holds. Clearly, $\hat{\sigma}_{t,F}[x_i] = x_i$ and we see that $\hat{\sigma}_{t,F}[\gamma(x_{\phi(1)}, x_{\phi(2)})] = \hat{\sigma}_{t,F}[\gamma(x_1, x_2)] = R^{lin}{}^2_2(\gamma(x_1, x_2), x_1, x_2) = \gamma(x_1, x_2)$ and thus by Lemma 5.2 we get that $\sigma_{t,F}$is idempotent.

Now, it comes to characterize the idempotent element in $B_5, \ldots, B_8$. We first show that all elements in B_8 are not idempotent and then show that the idempotency of B_5, B_6, B_7 need the some conditions. In fact, we have the following results.

Theorem 5.7. Every $\sigma_{t,F} \in B_8$ is not idempotent.

Proof. Let $\sigma_{t,F} \in B_8$ with $t = f(x_{\pi(1)}, \dots, x_{\pi(n)}), F = \neg\gamma(x_{\phi(1)}, x_{\phi(2)})$. Suppose the contrary that $\sigma_{t,F}$ is idempotent, by Lemma 5.2, we obtain that $\hat{\sigma}_{t,F}[t] = t$ and $\hat{\sigma}_{t,F}[F] = F$. Obviously, $\hat{\sigma}_{t,F}[\neg\gamma(x_{\phi(1)}, x_{\phi(2)})] \neq \neg\gamma(x_{\phi(1)}, x_{\phi(2)})$ since we have already shown this inequality holds in Theorem 5.4 (iii). It contradicts to the result of our assumption. Therefore, $\sigma_{t,F}$ is not idempotent.

Next, we show that there is an element in B_5 which is not idempotent as the following example.

Example 5.8. Let ((3), (2)) be a type, i.e., we have one ternary operation symbol and one binary relation symbol, say f and γ, respectively. If we consider $\sigma_{t,F} \in B_5$ with $t = f(x_3, x_1, x_2)$ and $F = x_1 \approx x_2$., then we have $\hat{\sigma}_{t,F}[f(x_3, x_1, x_2)]$ $= S^{lin}{}^3_3(f(x_3, x_1, x_2), x_3, x_1, x_2) = f(x_2, x_3, x_1)$. By Lemma 5.2, we conclude that $\sigma_{t,F}$ is not idempotent.

We remark here that if we let $\sigma_{t,F} \in B_5,\dots,B_8$, then $\hat{\sigma}_{t,F}[F]$ has the same situation in the previous theorems. So, we are interesting in the way to find some conditions for the idempotency of $\hat{\sigma}_{t,F}[t]$.The next theorem shows that if we set some conditions, then we get the characterization of idempotent elements in B_5, B_6, B_7.

Theorem 5.9. Let$\sigma_{t,F} \in Hyp^{lin}((n),(2))$. Then the following statements hold.

(i) $\sigma_{t,F} \in B_5$ is idempotent if and only if $\pi(i) = i$ for all $i = 1, \dots, n$.

(ii) $\sigma_{t,F} \in B_6$is idempotent if and only if $\pi(i) = i$ for all $i = 1, \dots, n$ and $\phi(j) = j$ for all $j = 1,2$.

(iii) $\sigma_{t,F} \in B_7$ is idempotent if and only if $\pi(i) = i$ for all $i = 1, \dots, n$.

Proof. (i) Let $\sigma_{t,F} \in B_5$with$t = f(x_{\pi(1)}, \dots, x_{\pi(n)})$ and $F = x_{\phi(1)} \approx x_{\phi(2)}$. Now we may assume that if $\pi(i) \neq i$ for some $i = 1, \dots, n$. Then $\hat{\sigma}_{t,F}[f(x_{\pi(1)}, \dots, x_{\pi(n)})] = S^{lin}{}^n_n(f(x_{\pi(1)}, \dots, x_{\pi(n)}), x_{\pi(1)}, \dots, x_{\pi(n)}) = f(x_{\pi(\pi(1))}, \dots, x_{\pi(\pi(n))})$. By our assumption, $f(x_{\pi(\pi(1))}, \dots, x_{\pi(\pi(n))}) \neq f(x_{\pi(1)}, \dots, x_{\pi(n)})$ and thus $\sigma_{t,F}$is not idempotent. Conversely, assume that the condition holds. To show that $\sigma_{t,F}$ is idempotent we consider $\hat{\sigma}_{t,F}[f(x_{\pi(1)}, \dots, x_{\pi(n)})] = \hat{\sigma}_{t,F}[f(x_1, \dots, x_n)] = f(x_1, \dots, x_n)$ so that $\hat{\sigma}_{t,F}[t] = t$. We can prove similarly to the proof of Theorem 5.4(i) that $\hat{\sigma}_{t,F}[F] = F$. Therefore, $\sigma_{t,F}$ is idempotent.

(ii) Let $\sigma_{t,F} \in B_6$ with $= f(x_{\pi(1)}, \dots, x_{\pi(n)})$ and $F = \gamma(x_{\phi(1)}, x_{\phi(2)})$. We first assume that $\pi(i) \neq i$ for some $i = 1, \dots, n$ or $\phi(j) \neq j$ for some $j = 1,2$. Then by the same manner as in the proof of (i) we can show that $\sigma_{t,F}$ is not idempotent. Conversely, assume that the condition holds. Clearly, $\hat{\sigma}_{t,F}[f(x_{\pi(1)}, \dots, x_{\pi(n)})] = \hat{\sigma}_{t,F}[f(x_1, \dots, x_n)] = f(x_1, \dots, x_n)$ and thus $\hat{\sigma}_{t,F}[t] = t$. Moreover, we have that $\hat{\sigma}_{t,F}[\gamma(x_{\phi(1)}, x_{\phi(2)})] = \hat{\sigma}_{t,F}[\gamma(x_1, x_2)] = \gamma(x_1, x_2)$, that is $\hat{\sigma}_{t,F}[F] = F$. By Lemma 5.2, $\sigma_{t,F}$ is idempotent.

(iii) By using Lemma 5.1, we can prove similarly to the proof of (i) that this statement holds.

Note that every idempotent element is regular. We characterize all regular elements in $Hyp^{lin}((n),(2))$, we consider $\sigma_{t,F} \in Hyp^{lin}((n),(2))$ which is not idempotent. The characterization of regularity in $Hyp^{lin}((n),(2))$ can be shown in the next theorem.

Theorem 5.10. Let $\sigma_{t,F} \in Hyp^{lin}((n),(2))$. Then the following statements hold.

(i) Every $\sigma_{t,F} \in B_2$ is regular.

(ii) Every $\sigma_{t,F} \in B_4$ is regular.

(iii) Every $\sigma_{t,F} \in B_5$ is regular.

(iv) Every $\sigma_{t,F} \in B_6$ is regular.

(v) Every $\sigma_{t,F} \in B_7$ is regular.

(vi) Every $\sigma_{t,F} \in B_8$ is regular.

Proof. (i) Let $\sigma_{t,F} \in B_2$ with$t = x_i$ and $F = \gamma(x_{\phi(1)}, x_{\phi(2)})$. We consider regularity of $\sigma_{t,F} \in B_2$ only the case of $\phi(j) \neq j$ for some $j = 1,2$. To do this, we choose $\sigma_{\acute{t},\acute{F}} \in B_2$ with $\acute{t} = x_i$ and $\acute{F} = \gamma(x_{\phi^{-1}(1)}, x_{\phi^{-1}(2)})$ such that $(\sigma_{t,F} \circ_r \sigma_{\acute{t},\acute{F}} \circ_r \sigma_{t,F})(f) = x_i = \sigma_{t,F}(f)$ and $(\sigma_{t,F} \circ_r \sigma_{\acute{t},\acute{F}} \circ_r \sigma_{t,F})(\gamma) = \hat{\sigma}_{t,F}[\hat{\sigma}_{\acute{t},\acute{F}}[\gamma(x_{\phi(1)}, x_{\phi(2)})]] = \hat{\sigma}_{t,F}[R^{lin}{}^2_2(\sigma_{\acute{t},\acute{F}}(\gamma), x_{\phi(1)}, x_{\phi(2)})] = \hat{\sigma}_{t,F}[R^{lin}{}^2_2(\gamma(x_{\phi^{-1}(1)}, x_{\phi^{-1}(2)}), x_{\phi(1)}, x_{\phi(2)})] = \hat{\sigma}_{t,F}\left[\gamma(x_{\phi(\phi^{-1}(1))}, x_{\phi(\phi^{-1}(2))})\right] = \hat{\sigma}_{t,F}[\gamma(x_{(\phi\circ\phi^{-1})(1)}, x_{(\phi\circ\phi^{-1})(2)})] = \hat{\sigma}_{t,F}[\gamma(x_1, x_2)] = R^{lin}{}^2_2(\gamma(x_{\phi(1)}, x_{\phi(2)}), x_1, x_2) = \gamma(x_{\phi(1)}, x_{\phi(2)}) = \sigma_{t,F}(\gamma)$. This implies that, $\sigma_{t,F}$ is regular.

(ii) Similarly to the proof of (i) and by using Lemma 5.1, we can show that every $\sigma_{t,F} \in B_4$ is regular.

(iii) Let $\sigma_{t,F} \in B_5$ with $t = f(x_{\pi(1)}, \dots, x_{\pi(n)})$ and $F = x_{\phi(1)} \approx x_{\phi(2)}$. We consider in the case of $\pi(i) \neq i$ for some $i = 1, \dots, n$, then there exists $\sigma_{\acute{t},\acute{F}} \in B_2$ with $\acute{t} = f(x_{\pi^{-1}(1)}, \dots, x_{\pi^{-1}(n)})$ and $\acute{F} = x_{\phi(1)} \approx x_{\phi(2)}$ such that $(\sigma_{t,F} \circ_r \sigma_{\acute{t},\acute{F}} \circ_r \sigma_{t,F})(f)$

$$
\begin{aligned}
&= \hat{\sigma}_{t,F}\left[\hat{\sigma}_{\acute{t},\acute{F}}[f(x_{\pi(1)}, \dots, x_{\pi(n)})]\right] \\
&= \hat{\sigma}_{t,F}[S^{lin}{}^n_n(\sigma_{\acute{t},\acute{F}}(f), x_{\pi(1)}, \dots, x_{\pi(n)})] \\
&= \hat{\sigma}_{t,F}[S^{lin}{}^n_n(f(x_{\pi^{-1}(1)}, \dots, x_{\pi^{-1}(n)}), x_{\pi(1)}, \dots, x_{\pi(n)})] \\
&= \hat{\sigma}_{t,F}\left[f\left(x_{\pi(\pi^{-1}(1))}, \dots, x_{\pi(\pi^{-1}(n))}\right)\right] \\
&= \hat{\sigma}_{t,F}[f(x_{\pi\circ\pi^{-1}(1)}, \dots, x_{\pi\circ\pi^{-1}(n)})] \\
&= \hat{\sigma}_{t,F}[f(x_1, \dots, x_n)] \\
&= S^{lin}{}^n_n(f(x_{\pi(1)}, \dots, x_{\pi(n)}), x_1, \dots, x_n) \\
&= f(x_{\pi(1)}, \dots, x_{\pi(n)}) \\
&= \sigma_{t,F}(f).
\end{aligned}
$$

And $(\sigma_{t,F} \circ_r \sigma_{\acute{t},\acute{F}} \circ_r \sigma_{t,F})(\gamma) = \hat{\sigma}_{t,F}[\hat{\sigma}_{\acute{t},\acute{F}}[x_{\phi(1)} \approx x_{\phi(2)}]] = \hat{\sigma}_{t,F}\left[\hat{\sigma}_{\acute{t},\acute{F}}[x_{\phi(1)}] \approx \hat{\sigma}_{\acute{t},\acute{F}}[x_{\phi(2)}]\right] = \hat{\sigma}_{t,F}[x_{\phi(1)} \approx x_{\phi(2)}] = x_{\phi(1)} \approx x_{\phi(2)}$.

(iv) Let $\sigma_{t,F} \in B_6$ with $t = f(x_{\pi(1)}, \dots, x_{\pi(n)})$ and $F = \gamma(x_{\phi(1)}, x_{\phi(2)})$.

To prove that $\sigma_{t,F}$ is regular, we consider in three cases: If $\pi(i) = i$ for all $i = 1, \dots, n$ and $\phi(j) \neq j$ for some $j = 1,2$, then there exists $\sigma_{\acute{t},\acute{F}} \in B_6$ with $\acute{t} = f(x_{\pi(1)}, \dots, x_{\pi(n)})$ and $F\acute{} = \gamma(x_{\phi^{-1}(1)}, x_{\phi^{-1}(2)})$ such that $(\sigma_{t,F} \circ_r \sigma_{\acute{t},\acute{F}} \circ_r \sigma_{t,F})(f)$

$$
\begin{aligned}
&= \hat{\sigma}_{t,F}[\hat{\sigma}_{\acute{t},\acute{F}}[f(x_{\pi(1)}, \dots, x_{\pi(n)})]] \\
&= \hat{\sigma}_{t,F}[S^{lin}{}^n_n(f(x_{\pi(1)}, \dots, x_{\pi(n)}), x_{\pi(1)}, \dots, x_{\pi(n)})] \\
&= \hat{\sigma}_{t,F}[f\left(x_{\pi(\pi(1))}, \dots, x_{\pi(\pi(n))}\right)] \\
&= \hat{\sigma}_{t,F}[f(x_{\pi(1)}, \dots, x_{\pi(n)})] \\
&= S^{lin}{}^n_n(f(x_{\pi(1)}, \dots, x_{\pi(n)}), x_{\pi(1)}, \dots, x_{\pi(n)}) \\
&= f\left(x_{\pi(\pi(1))}, \dots, x_{\pi(\pi(n))}\right) \\
&= f(x_{\pi(1)}, \dots, x_{\pi(n)}) \\
&= \sigma_{t,F}(f).
\end{aligned}
$$

Similarly to the proof of (i), we have that $(\sigma_{t,F} \circ_r \sigma_{\acute{t},\acute{F}} \circ_r \sigma_{t,F})(\gamma) = \sigma_{t,F}(\gamma)$.

If $\pi(i) \neq i$ for some $i = 1, \dots, n$ and $\phi(j) = j$ for all $j = 1,2$, then there exists $\sigma_{\acute{t},\acute{F}} \in B_6$ with $\acute{t} = f(x_{\pi^{-1}(1)}, \dots, x_{\pi^{-1}(n)})$and $F\acute{} = \gamma(x_{\phi(1)}, x_{\phi(2)})$ such that $(\sigma_{t,F} \circ_r \sigma_{\acute{t},\acute{F}} \circ_r \sigma_{t,F})(f) = \sigma_{t,F}(f)$, it follows from (iii). Moreover, we consider $(\sigma_{t,F} \circ_r \sigma_{\acute{t},\acute{F}} \circ_r \sigma_{t,F})(\gamma)$

$= \hat{\sigma}_{t,F}[\hat{\sigma}_{\acute{t},\acute{F}}[\gamma(x_{\phi(1)}, x_{\phi(2)})]]$

$=\hat{\sigma}_{t,F}[S^{lin}{}^{2}_{2}(\gamma(x_{\phi(1)}, x_{\phi(2)}), x_{\phi(1)}, x_{\phi(2)})]$

$= \hat{\sigma}_{t,F}[\gamma\left(x_{\phi(\phi(1))}, x_{\phi(\phi(1))}\right)]$

$= \hat{\sigma}_{t,F}[\gamma(x_{\phi(1)}, x_{\phi(2)})]$

$= S^{lin}{}^{2}_{2}(\gamma(x_{\phi(1)}, x_{\phi(2)}), x_{\phi(1)}, x_{\phi(2)})$

$= \gamma\left(x_{\phi(\phi(1))}, x_{\phi(\phi(2))}\right)$

$= \gamma(x_{\phi(1)}, x_{\phi(2)})$

$= \sigma_{t,F}(\gamma).$

Finally, if $\pi(i) \neq i$ for some $i = 1, \ldots, n$and $\phi(j) \neq j$ for some $j = 1,2$, then there exists $\sigma_{\acute{t},\acute{F}} \in B_6$ with $\acute{t} = f(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)})$ and $F\acute{} = \gamma(x_{\phi^{-1}(1)}, x_{\phi^{-1}(2)})$ such that $(\sigma_{t,F} \circ_r \sigma_{\acute{t},\acute{F}} \circ_r \sigma_{t,F})(f) = \sigma_{t,F}(f)$ and $(\sigma_{t,F} \circ_r \sigma_{\acute{t},\acute{F}} \circ_r \sigma_{t,F})$ $(\gamma) = \sigma_{t,F}(\gamma)$. Therefore, we conclude that $\sigma_{t,F}$ is regular.

(v) This statement can be proved by using Lemma 5.1 and the same process as we proved in (iii).

(vi) This statement can be proved by using Lemma 5.1 and the same process as we proved in (iv).

Consequence of this section, every linear hypersubstitution is regular and then $Hyp^{lin}((n), (2))$ is a regular semigroup.

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