Songklanakarin J. Sci. Technol.
42 (1), 81-94, Jan. - Feb. 2020

http://www.sjst.psu.ac.th

Original Article

New transform formulae for differential transformation method with applications to the nonlinear plane autonomous systems

Uraiwan Somboon[1*], Jaipong Kasemsuwan[1], and Sorin V. Sabau[2]

[1] *Department of Mathematics, Faculty of Science, King Mongkut's Institute of Technology Ladkrabang, Lat Krabang, Bangkok, 10520 Thailand*

[2] *Department of Mathematics, Tokai University, Sapporo Campus, Minamisawa, Minamiku, Sapporo, 0058601 Japan*

Received: 3 August 2018; Accepted: 24 September 2018

Abstract

This work presents a new derivation technique for new differential transform formulae of a product of composite functions. The new formulae are applied to nonlinear plane autonomous systems to demonstrate their efficiency and reliability. The approximate series solutions estimated by the differential transform method (DTM) and the multistep differential transform method (MsDTM) are then compared with the flow direction of the vector fields defined by the original system and an analytical solution calculated by the phase-plane method. We found that the MsDTM results are in better agreement with the analytical solution than the DTM ones. Moreover, the MsDTM can be applied to systems whose analytical solutions are unobtainable. The approximate solutions by the MsDTM have the same direction to the flow of the vector field of the system. It follows that the proposed new formulae are reliable and efficient.

Keywords: differential transform method, nonlinear plane autonomous systems, multistep differential transform method, phase-plane method

1. Introduction

Autonomous systems are systems of first-order differential equations of the form

$$\frac{dx_1}{dt} = g_1(\mathrm{x}_1,\ldots,\mathrm{x}_n)$$
$$\frac{dx_2}{dt} = g_2(\mathrm{x}_1,\ldots,\mathrm{x}_n)$$
$$\vdots$$
$$\frac{dx_n}{dt} = g_n(\mathrm{x}_1,\ldots,\mathrm{x}_n)$$

such that the independent variable does not explicitly appear on the right hand side of each differential equation. In the case of $n = 2$, the system is called a plane autonomous system and $V(\mathrm{x}_1,\mathrm{x}_2) = \left(g_1(\mathrm{x}_1,\mathrm{x}_2), \mathrm{g}_2(\mathrm{x}_1,\mathrm{x}_2)\right)$ is a vector field in the plane that indicates the movement direction. If the parameter t is interpreted as time, then $X(\mathrm{t}) = \left(x(\mathrm{t}), \mathrm{y}(\mathrm{t})\right)$ indicates the position of the particle in the plane at time t and a solution of the system is interpreted as a path of this particle starting from $X(0,0) = \left(x(0), \mathrm{y}(0)\right)$ (Zill & Wright, 2014).

The differential transformation method (DTM) is an alternative procedure for obtaining an approximate Taylor series solution of differential equations. The main advantage of this method is that it can be applied directly to nonlinear differential equations without requiring linearization and discretization. The concept of the differential transform method was introduced by Zhou (Zhou, 1986), who solved linear and nonlinear problems in electrical circuits and many

*Corresponding author
Email address: uraiwan.so@hotmail.com

other problems related to differential equations (Damirchi & Shamami, 2016; Mahgoub & Alshikh, 2017; Methi, 2016; Mirzaee, 2011; Moon, Bhosale, Gajbhiye, & Lonare, 2014; Patil & Khambayat, 2014).

Although the DTM series solution gives a good approximation for some problems, in some cases, the series solution diverges in a wider domain. Due to this reason the multistep differential transform method (MsDTM) is used. The MsDTM is based on the DTM, but compared with other methods it does not need small parameters, auxiliary functions and parameters, or discretization. In this technique, the solution domain is divided in subdomains (Ebenezer, Freihet, Khan & Khan, 2016; Ertürk, Odibat & Momami, 2012; Odibat, Bertelle, Aziz-Alaoui & Duchamp, 2010; Rashidi, Chamkha, & Keimanesh, 2011; Zurigat & Ababneh, 2015). In particular, we are interested in the technique introduced by Chang (Change & Chang, 2008), to calculate the DTM of nonlinear functions.

In this paper, a derivation technique of new differential transform formulae for the product of composite functions is presented. The computation consists of three steps. The first step is finding the differential transformation of the product of two composite functions in the general form as shown in Equation 3.1. The next step is finding the differential transformation for the higher order derivative of a power function as shown in Lemma 3.1. The last step is the derivation of the new differential transformation shown in Formulae 1–8, calculated by using the general formulae of higher order derivatives of composite functions studied in (Weisstein, n.d.) combined with Lemma 3.1. Then, the new differential transform formulae obtained are used to transform the nonlinear plane autonomous systems to find the DTM and MsDTM approximate solutions of the problem. By comparing graphically the results, we obtain the approximate series solutions calculated by the DTM and the MsDTM that have the same direction with the vector fields flow and they are also similar to the analytical solution obtained by the phase-plane method.

Here is the structure of the paper. In section 2, the one-dimensional differential transformation method is described. In section 3, the analysis of the method and new formulae calculation are proposed. In section 4, the new differential transform formulae proposed are applied to three examples of nonlinear plane autonomous systems to show the reliability and efficiency of the method. The conclusion is given at the end of the paper in section 5.

2. Basic Definitions and Fundamental Operations of the One-Dimensional Differential Transform Method

2.1 Definition

The one-dimensional differential transform of the function $x(\mathrm{t})$ is defined as

$$X(\mathrm{k}) = \frac{1}{k!}\left[\frac{d^k x(\mathrm{t})}{dt^k}\right]_{t=t_0}, \; k \geq 0. \tag{2.1}$$

In Equation 2.1, $x(\mathrm{t})$ is called the original function and $X(\mathrm{k})$ is called the transformed function.

2.2 Definition

The inverse one-dimensional differential transform of $X(\mathrm{k})$ is defined as

$$x(\mathrm{t}) = \sum_{k=0}^{\infty} X(\mathrm{k})\left(\mathrm{t}-\mathrm{t}_0\right)^k, \tag{2.2}$$

that is,

$$x(\mathrm{t}) = \sum_{k=0}^{\infty} \frac{\left(t-t_0\right)^k}{k!}\left[\frac{d^k x(\mathrm{t})}{dt^k}\right]_{t=t_0}. \tag{2.3}$$

Equation 2.3 implies that the concept of differential transformation method is derived from Taylor series expansion. Actually, in concrete applications, the function $x(\mathrm{t})$ is expressed by a truncated series and Equation 2.2 becomes

$$\mathrm{x}(\mathrm{t}) = \sum_{k=0}^{N} X(\mathrm{k})\left(\mathrm{t}-\mathrm{t}_0\right)^k. \tag{2.4}$$

2.3 Fundamental operations

The fundamental operations of the one-dimensional DTM are shown in Table 1. The multistep differential transformation method (MsDTM) is advantageous for applications in physics. For instance, due to small time steps the MsDTM has a powerful accuracy especially for an initial value problem (IVP).

Let $[0,\mathrm{T}]$ be the interval over which we want to find the solution of the IVP. In actual applications of the DTM, the approximate solution of the IVP can be expressed by the finite series

$$\mathrm{x}(\mathrm{t}) = \sum_{k=0}^{N} X(\mathrm{k})\left(\mathrm{t}\right)^k, \; \mathrm{t} \in [0,\mathrm{T}]. \tag{2.5}$$

Let us assume that the interval $[0,\mathrm{T}]$ in divided into n subintervals $[\mathrm{t}_{i-1},\mathrm{t}_i]$, $\mathrm{i}=1,\ldots,\mathrm{m}$ of equal step size $h = T/m$ by using the nodes $\mathrm{t}_i = ih.$ The main ideas of the MsDTM can be found in (Odibat, Bertelle, Aziz-Alaoui, & Duchamp, 2010). In fact, the MsDTM gives the solution in the form,

$$x(\mathrm{t}) = \begin{cases} \mathrm{x}_0(\mathrm{t}), & t \in [0,\mathrm{t}_1] \\ \mathrm{x}_1(\mathrm{t}), & t \in [\mathrm{t}_1,\mathrm{t}_2] \\ \vdots & \\ \mathrm{x}_m(\mathrm{t}), & t \in [t_m,\mathrm{t}_{m-1}], \end{cases} \tag{2.6}$$

Table 1. Fundamental operations of one-dimensional DTM.

Original function $x(t)$	Transformed functions $X(k)$
$x(t) \pm y(t)$	$X(k) \pm Y(k)$
$\lambda x(t)$	$\lambda X(k)$
$x(t)\, y(t)$	$\sum_{r=0}^{k} X(r)\, Y(k-r)$
$x(t)\, y(t)\, z(t)$	$\sum_{r=0}^{k}\sum_{l=0}^{r} X(l)\, Y(r-l)\, Z(k-r)$
$\frac{d^r}{dt^r} x(t)$	$\frac{(k+r)!}{k!} X(k+r)$

where $x_i(t) = \sum_{k=0}^{N} X_i(k)(t-t_i)^k$ and the initial condition $x_i^{(k)}(t_{i-1}) = x_{i-1}^{(k)}(t_{i-1})$.

3. Analysis of Method

This section introduces our derivation technique of the new differential transform formulae for the product of composite functions derived in Formulae 1–8. To obtain these new formulae, the derivation is shown in the following steps.

Step 1. The differential transformation for the product of two composite functions is represented by $f(y(t))g(y(t))$, which are the original functions. By the definition given in Section 2.1 of the DTM combined with Leibniz formula, we obtain

$$\frac{1}{k!}\left[\frac{d^k}{dt^k} f(y(t))\, g(y(t))\right] = \frac{1}{k!}\left[\sum_{r=0}^{k}\frac{k!}{(k-r)!r!}\frac{d^r}{dt^r} f(y(t))\frac{d^{k-r}}{dt^{k-r}} g(y(t))\right]_{t=t_0}$$
$$= \sum_{r=0}^{k} F(r)\, G(k-r), \tag{3.1}$$

where $F(r) = \frac{1}{r!}\left[\frac{d^r}{dt^r} f(y(t))\right]_{t=t_0}$, $G(k-r) = \frac{1}{(k-r)!}\left[\frac{d^{k-r}}{dt^{k-r}} g(y(t))\right]_{t=t_0}$.

Step 2. This step finds the differential transformation for the higher order derivative of the power function that is used in Step 3.

Lemma 3.1. If $k, r, m \in I^+ \cup \{0\}$ and let $w = r - m = 0,\dots,r$ where $r = 0,\dots,k,\ m = 0,\dots,r$, then

$$\left[\frac{1}{k!}\frac{d^k y(t)^w}{dt^k}\right]_{t=t_0} = 1, \quad k = 0,$$

$$\left[\frac{1}{k!}\frac{d^k y(t)^w}{dt^k}\right]_{t=t_0} = \sum_{k_{w-1}=0}^{k}\sum_{k_{w-2}=0}^{k_{w-1}}\cdots\sum_{k_1=0}^{k_2} Y(k_1)Y(k_2-k_1)\cdots Y(k-k_{w-1}), \quad k>0. \tag{3.2}$$

Proof. Assume that $k, r, m \in I^+ \cup \{0\}$ and let $w = r - m = 0,\dots,r$ where $r = 0,\dots,k,\ m = 0,\dots,r$

Case $k = 0$; we have $r = 0$ and $m = 0$, then $\left[\frac{1}{0!}\frac{d^0 y(t)^0}{dt^0}\right]_{t=t_0} = 1.$

Case $k>0;$ we will prove by mathematical induction. Let $P(\mathrm{w})$ be Equation 3.2.
First, we will show that the statement holds for $w=0$, that is

$$P(0)=\left[\frac{1}{k!}\frac{d^k y(\mathrm{t})^0}{dt^k}\right]_{t=t_0}=0.$$

Next, we assume that the statement is true for $w=r-1$, that is

$$P(\mathrm{r}-1)=\left[\frac{1}{k!}\frac{d^k y(\mathrm{t})^{r-1}}{dt^k}\right]_{t=t_0}=\sum_{k_{r-2}=0}^{k}\sum_{k_{r-3}=0}^{k_{r-2}}\cdots\sum_{k_1=0}^{k_2}Y(\mathrm{k}_1)Y(\mathrm{k}_2-\mathrm{k}_1)\cdots\mathrm{Y}(\mathrm{k}-\mathrm{k}_{r-2}).$$

We will show that the statement is also true for $w=r$. This can be seen as follows

$$\begin{aligned}
P(\mathrm{r})&=\left[\frac{1}{k!}\frac{d^k y(\mathrm{t})^r}{dt^k}\right]_{t=t_0}=\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})^{r-1}y(\mathrm{t})\right)\right]\\
&=\left[\frac{1}{k!}\sum_{k_{r-1}=0}^{k}\frac{k!}{(\mathrm{k}-\mathrm{k}_{r-1})!\mathrm{k}_{r-1}!}\frac{d^{k_{r-1}}}{dt^{k_{r-1}}}y(\mathrm{t})^{r-1}\frac{d^{k-k_{r-1}}}{dt^{k-k_{r-1}}}y(\mathrm{t})\right]_{t=t_0}\\
&=\sum_{k_{r-1}=0}^{k}\left[\frac{1}{\mathrm{k}_{r-1}!}\frac{d^{k_{r-1}}}{dt^{k_{r-1}}}y(\mathrm{t})^{r-1}\frac{1}{(\mathrm{k}-\mathrm{k}_{r-1})!}\frac{d^{k-k_{r-1}}}{dt^{k-k_{r-1}}}y(\mathrm{t})\right]_{t=t_0}\\
&=\sum_{k_{r-1}=0}^{k}\sum_{k_{r-2}=0}^{k_{r-1}}\cdots\sum_{k_1=0}^{k_2}Y(\mathrm{k}_1)Y(\mathrm{k}_2-\mathrm{k}_1)\cdots\mathrm{Y}(\mathrm{k}-\mathrm{k}_{r-1}).
\end{aligned}$$

Therefore, the statement holds for $w=r,$ and the proof is completed.

Step 3. The functions $f\left(y(\mathrm{t})\right)$ and $g\left(y(\mathrm{t})\right)$ in Step 1 are considered as the original functions in the Formulae 1–8. To obtain these new differential transform formulae, the general formulae of higher order derivatives of some composite functions are used together with Lemma 3.1 in the following calculations.

Formula 1. If $f\left(y(\mathrm{t})\right)=e^{y(\mathrm{t})}$ is the original function, then

$$\begin{aligned}
F(\mathrm{k})&=\frac{1}{k!}\left[\frac{d^k}{dt^k}e^{y(\mathrm{t})}\right]_{t=t_0}=\frac{1}{k!}\left[e^{y(\mathrm{t})}\sum_{r=0}^{k}\frac{1}{r!}\sum_{m=0}^{r}\frac{(-1)^m r!}{(\mathrm{r}-\mathrm{m})!\mathrm{m}!}y^m(\mathrm{t})\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_0}\\
&=\mathrm{e}^{y(\mathrm{t}_0)}\sum_{r=0}^{k}\sum_{m=0}^{r}\frac{(-1)^m y^m(\mathrm{t}_0)}{(\mathrm{r}-\mathrm{m})!\mathrm{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_0}\\
&=\mathrm{e}^{Y(0)}\sum_{r=0}^{k}\sum_{m=0}^{r}\frac{(-1)^m Y^m(0)}{(\mathrm{r}-\mathrm{m})!\mathrm{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_0},
\end{aligned}$$

where $Y(0)=y(\mathrm{t}_0)$, and we have used Lemma 3.1 to transform $\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_0}$.

Formula 2. If $f\left(y(\mathrm{t})\right)=\ln\left(y(\mathrm{t})\right),\ \mathrm{y}(\mathrm{t})>0$ is the original function, then

$$\begin{aligned}
F(\mathrm{k})&=\frac{1}{k!}\left[\frac{d^k}{dt^k}\ln\left(y(\mathrm{t})\right)\right]_{t=t_0}=\frac{1}{k!}\left[\delta_k\ln\left(y(\mathrm{t})\right)+\sum_{r=1}^{k}\frac{(-1)^{r-1}}{ry^r(\mathrm{t})}\binom{k}{r}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^r\right]_{t=t_0}\\
&=\frac{1}{k!}\delta_k\ln\left(Y(0)\right)+\sum_{r=1}^{k}\frac{(-1)^{r-1}}{rY^r(\mathrm{t}_0)}\binom{k}{r}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^r\right]_{t=t_0},
\end{aligned}$$

where $\binom{k}{r} = \dfrac{\text{k!}}{(k-r)!\text{r!}}$ are the binomial coefficients, $\delta_k = \begin{cases} 1, & k=0 \\ 0, & k=1,2,3,\ldots \end{cases}$ and

$$\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\text{t})\right)^r\right]_{t=t_0} = \sum_{k_{r-1}=0}^{k}\sum_{k_{r-2}=0}^{k_{r-1}}\ldots\sum_{k_1=0}^{k_2} Y(\text{k}_1)Y(\text{k}_2-\text{k}_1)\ldots\text{Y}(\text{k}-\text{k}_{r-1})\cdot$$

Formula 3. If $f\left(y(\text{t})\right)=\sin\left(y(\text{t})\right)$ is the original function, then

$$F(\text{k}) = \frac{1}{k!}\left[\frac{d^k}{dt^k}\sin\left(y(\text{t})\right)\right]_{t=t_0} = \frac{1}{k!}\left[\sum_{r=0}^{k}\frac{1}{r!}\frac{d^r}{dt^r}\sin(\text{t})\bigg|_{t=y(\text{t})}\left(\sum_{m=0}^{r}\frac{(-1)^m r!}{(\text{r}-\text{m})!\text{m}!}y^m(\text{t})\frac{d^k}{dt^k}\left(y(\text{t})\right)^{r-m}\right)\right]_{t=t_0}$$

$$= \sum_{r=0}^{k}\frac{d^r}{dt^r}\sin(\text{t})\bigg|_{t=y(\text{t}_0)}\sum_{m=0}^{r}\frac{(-1)^m y^m(\text{t}_0)}{(\text{r}-\text{m})!\text{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\text{t})\right)^{r-m}\right]_{t=t_0}$$

$$= \sum_{r=0}^{k}\frac{d^r}{dt^r}\sin(\text{t})\bigg|_{t=Y(0)}\sum_{m=0}^{r}\frac{(-1)^m Y^m(0)}{(\text{r}-\text{m})!\text{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\text{t})\right)^{r-m}\right]_{t=t_0}$$

where $Y(0)=y(\text{t}_0)$, and we have used Lemma 3.1 to transform $\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\text{t})\right)^{r-m}\right]_{t=t_0}$.

Formula 4. If $f\left(y(\text{t})\right)=\cos\left(y(\text{t})\right)$ is the original function, then

$$F(\text{k}) = \frac{1}{k!}\left[\frac{d^k}{dt^k}\cos\left(y(\text{t})\right)\right]_{t=t_0} = \frac{1}{k!}\left[\sum_{r=0}^{k}\frac{1}{r!}\frac{d^r}{dt^r}\cos(\text{t})\bigg|_{t=y(\text{t})}\left(\sum_{m=0}^{r}\frac{(-1)^m r!}{(\text{r}-\text{m})!\text{m}!}y^m(\text{t})\frac{d^k}{dt^k}\left(y(\text{t})\right)^{r-m}\right)\right]_{t=t_0}$$

$$= \sum_{r=0}^{k}\frac{d^r}{dt^r}\cos(\text{t})\bigg|_{t=y(\text{t}_0)}\sum_{m=0}^{r}\frac{(-1)^m y^m(\text{t}_0)}{(\text{r}-\text{m})!\text{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\text{t})\right)^{r-m}\right]_{t=t_0}$$

$$= \sum_{r=0}^{k}\frac{d^r}{dt^r}\cos(\text{t})\bigg|_{t=Y(0)}\sum_{m=0}^{r}\frac{(-1)^m Y^m(0)}{(\text{r}-\text{m})!\text{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\text{t})\right)^{r-m}\right]_{t=t_0}$$

where $Y(0)=y(\text{t}_0)$, and we have used Lemma 3.1 to transform $\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\text{t})\right)^{r-m}\right]_{t=t_0}$.

Formula 5. If $f\left(y(\text{t})\right)=\sinh\left(y(\text{t})\right)$ is the original function, then

$$F(\text{k}) = \frac{1}{k!}\left[\frac{d^k}{dt^k}\sinh\left(y(\text{t})\right)\right]_{t=t_0} = \frac{1}{k!}\left[\sum_{r=0}^{k}\frac{1}{r!}\frac{d^r}{dt^r}\sinh(\text{t})\bigg|_{t=y(\text{t})}\left(\sum_{m=0}^{r}\frac{(-1)^m r!}{(\text{r}-\text{m})!\text{m}!}y^m(\text{t})\frac{d^k}{dt^k}\left(y(\text{t})\right)^{r-m}\right)\right]_{t=t_0}$$

$$= \sum_{r=0}^{k}\frac{d^r}{dt^r}\sinh(\text{t})\bigg|_{t=y(\text{t}_0)}\sum_{m=0}^{r}\frac{(-1)^m y^m(\text{t}_0)}{(\text{r}-\text{m})!\text{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\text{t})\right)^{r-m}\right]_{t=t_0}$$

$$= \sum_{r=0}^{k}\frac{d^r}{dt^r}\sinh(\text{t})\bigg|_{t=Y(0)}\sum_{m=0}^{r}\frac{(-1)^m Y^m(0)}{(\text{r}-\text{m})!\text{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\text{t})\right)^{r-m}\right]_{t=t_0}$$

where $Y(0)=y(\text{t}_0)$, and we have used 3.1 to transform $\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\text{t})\right)^{r-m}\right]_{t=t_0}$.

Formula 6. If $f\left(y(\mathrm{t})\right)=\cosh\left(y(\mathrm{t})\right)$ is the original function, then

$$F(\mathrm{k})=\frac{1}{k!}\left[\frac{d^k}{dt^k}\cosh\left(y(\mathrm{t})\right)\right]_{t=t_0}=\frac{1}{k!}\left[\sum_{r=0}^{k}\frac{1}{r!}\frac{d^r}{dt^r}\cosh(\mathrm{t})\bigg|_{t=y(\mathrm{t})}\left(\sum_{m=0}^{r}\frac{(-1)^m r!}{(\mathrm{r}-\mathrm{m})!\mathrm{m}!}y^m(\mathrm{t})\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right)\right]_{t=t_0}$$

$$=\sum_{r=0}^{k}\frac{d^r}{dt^r}\cosh(\mathrm{t})\bigg|_{t=y(\mathrm{t}_0)}\sum_{m=0}^{r}\frac{(-1)^m y^m(\mathrm{t}_0)}{(\mathrm{r}-\mathrm{m})!\mathrm{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_0}$$

$$=\sum_{r=0}^{k}\frac{d^r}{dt^r}\cosh(\mathrm{t})\bigg|_{t=Y(0)}\sum_{m=0}^{r}\frac{(-1)^m Y^m(0)}{(\mathrm{r}-\mathrm{m})!\mathrm{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_0}$$

where $Y(0)=y(\mathrm{t}_0)$, and we have used Lemma 3.1 to transform $\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_0}$.

Formula 7. If $f\left(y(\mathrm{t})\right)=\sqrt{y(\mathrm{t})}$ is the original function, then

$$F(\mathrm{k})=\frac{1}{k!}\left[\frac{d^k}{dt^k}\sqrt{y(\mathrm{t})}\right]_{t=t_0}=\frac{1}{k!}\left[\frac{\Gamma(\mathrm{k}+\frac{1}{2})}{2\Gamma(\mathrm{k}+1)\Gamma(\frac{1}{2})}\sum_{r=0}^{k}\frac{(-1)^r}{(\frac{1}{2}-\mathrm{r})}\binom{k}{r}\left(y(\mathrm{t})\right)^{\frac{1}{2}-r}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^r\right]_{t=t_0}$$

$$=\frac{\Gamma(\mathrm{k}+\frac{1}{2})}{2\Gamma(\mathrm{k}+1)\Gamma(\frac{1}{2})}\sum_{r=0}^{k}\frac{(-1)^r}{(\frac{1}{2}-\mathrm{r})}\binom{k}{r}\left(\mathrm{Y}(0)\right)^{\frac{1}{2}-r}\left[\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^r\right]_{t=t_0}$$

where $\binom{k}{r}=\frac{\mathrm{k}!}{\left(k-r\right)!\mathrm{r}!}$ are the binomial coefficients,

$\Gamma(1+\mathrm{z})=\mathrm{z}\,\Gamma(\mathrm{z}),\ \mathrm{z}\in\mathbb{Q},\ \ \Gamma(\tfrac{1}{2})=\sqrt{\pi},\ \Gamma(\mathrm{n})=(\mathrm{n}-1)!,\ \mathrm{n}\in\mathbb{Z}^+,$ and

$$\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^r\right]_{t=t_0}=\sum_{k_{r-1}=0}^{k}\sum_{k_{r-2}=0}^{k_{r-1}}\cdots\sum_{k_1=0}^{k_2}Y(\mathrm{k}_1)Y(\mathrm{k}_2-\mathrm{k}_1)\ldots\mathrm{Y}(\mathrm{k}-\mathrm{k}_{r-1}).$$

Formula 8. If $f\left(y(\mathrm{t})\right)=\frac{1}{y(\mathrm{t})}$ is the original function, then

$$F(\mathrm{k})=\frac{1}{k!}\left[\frac{d^k}{dt^k}\frac{1}{y(\mathrm{t})}\right]_{t=t_0}=\frac{1}{k!}\left[(\mathrm{k}+1)\sum_{r=0}^{k}\frac{(-1)^r}{(\mathrm{r}+1)}\binom{k}{r}\left(y(\mathrm{t})\right)^{-r-1}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^r\right]_{t=t_0}$$

$$=(\mathrm{k}+1)\sum_{r=0}^{k}\frac{(-1)^r}{(\mathrm{r}+1)}\binom{k}{r}\left(\mathrm{Y}(\mathrm{t}_0)\right)^{-r-1}\left[\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^r\right]_{t=t_0}$$

where $\binom{k}{r}=\frac{\mathrm{k}!}{\left(k-r\right)!\mathrm{r}!}$ are the binomial coefficients, and

$$\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^r\right]_{t=t_0}=\sum_{k_{r-1}=0}^{k}\sum_{k_{r-2}=0}^{k_{r-1}}\cdots\sum_{k_1=0}^{k_2}Y(\mathrm{k}_1)Y(\mathrm{k}_2-\mathrm{k}_1)\ldots\mathrm{Y}(\mathrm{k}-\mathrm{k}_{r-1}).$$

The transformed functions are shown in Table 2.

Table 2. Transformed functions of some nonlinear functions.

Original function $f\left(y(\mathrm{t})\right)$	Transformed functions $F(\mathrm{k})$
$e^{y(\mathrm{t})}$	$\mathrm{e}^{Y(\mathrm{t}_0)}\sum_{r=0}^{k}\sum_{m=0}^{r}\frac{(-1)^m Y^m(\mathrm{t}_0)}{(\mathrm{r-m})!\mathrm{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_0}$
$\ln\left(y(\mathrm{t})\right)$	$\frac{1}{k!}\delta_k \ln\left(Y(\mathrm{t}_0)\right)+\sum_{r=1}^{k}\frac{(-1)^{r-1}}{rY^r(\mathrm{t}_0)}\binom{k}{r}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r}\right]_{t=t_0}$
$\sin\left(\mathrm{y(t)}\right)$	$\sum_{r=0}^{k}\frac{d^r}{dt^r}\sin(\mathrm{t})\Big\vert_{t=Y(\mathrm{t}_0)}\sum_{m=0}^{r}\frac{(-1)^m Y^m(\mathrm{t}_0)}{(\mathrm{r-m})!\mathrm{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_0}$
$\cos\left(y(\mathrm{t})\right)$	$\sum_{r=0}^{k}\frac{d^r}{dt^r}\cos(\mathrm{t})\Big\vert_{t=Y(\mathrm{t}_0)}\sum_{m=0}^{r}\frac{(-1)^m Y^m(\mathrm{t}_0)}{(\mathrm{r-m})!\mathrm{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_0}$
$\sinh\left(y(\mathrm{t})\right)$	$\sum_{r=0}^{k}\frac{d^r}{dt^r}\sinh(\mathrm{t})\Big\vert_{t=Y(\mathrm{t}_0)}\sum_{m=0}^{r}\frac{(-1)^m Y^m(\mathrm{t}_0)}{(\mathrm{r-m})!\mathrm{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_0}$
$\cosh\left(y(\mathrm{t})\right)$	$\sum_{r=0}^{k}\frac{d^r}{dt^r}\cosh(\mathrm{t})\Big\vert_{t=Y(\mathrm{t}_0)}\sum_{m=0}^{r}\frac{(-1)^m Y^m(\mathrm{t}_0)}{(\mathrm{r-m})!\mathrm{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_0}$
$\sqrt{y(\mathrm{t})}$	$\frac{\Gamma(\mathrm{k}+\frac{1}{2})}{2\Gamma(\mathrm{k}+1)\Gamma(\frac{1}{2})}\sum_{r=0}^{k}\frac{(-1)^r}{(\frac{1}{2}-\mathrm{r})}\binom{k}{r}\left(\mathrm{Y(t_0)}\right)^{\frac{1}{2}-r}\left[\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r}\right]_{t=t_0}$
$\frac{1}{y(\mathrm{t})}$	$(\mathrm{k}+1)\sum_{r=0}^{k}\frac{(-1)^r}{(\mathrm{r}+1)}\binom{k}{r}\left(\mathrm{Y(t_0)}\right)^{-r-1}\left[\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r}\right]_{t=t_0}$

4. Applications

In this section, we extended the application of the DTM to nonlinear plane autonomous systems. To demonstrate the formulae introduced in the previous section, three examples are studied here. The accuracy of the method is assessed by graphical and data value comparisons.

Example 4.1. Consider the following system of nonlinear plane autonomous

$$x' = e^{y} \tag{4.1}$$

$$y' = e^{x},\ \text{for}\ t \in [0,1.25], \tag{4.2}$$

subject to the initial conditions $x(0)=0,\ \mathrm{y}(0)=0.$

Applying the DTM of Equations 4.1 and 4.2 and using the initial conditions $x(0)=0,\ \mathrm{y}(0)=0.$ it follows

$$\mathrm{X(k+1)} = \frac{1}{k+1}\left(\mathrm{e}^{Y(0)}\sum_{r=0}^{k}\sum_{m=0}^{r}\frac{(-1)^m Y^m(0)}{(\mathrm{r-m})!\mathrm{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=0}\right)$$

$$\mathrm{Y(k+1)} = \frac{1}{k+1}\left(\mathrm{e}^{\mathrm{X}(0)}\sum_{r=0}^{k}\sum_{m=0}^{r}\frac{(-1)^m X^m(0)}{(\mathrm{r-m})!\mathrm{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(\mathrm{x(t)}\right)^{r-m}\right]_{t=0}\right),$$

$$X(0)=0,\ \mathrm{Y}(0)=0.$$

By substituting $k=0,\ldots,11$ we obtain the coefficients of the series solution as follows

$$X(1)=\mathrm{Y}(1)=1,\ \mathrm{X}(2)=\mathrm{Y}(2)=\frac{1}{2},\ \mathrm{X}(3)=\mathrm{Y}(3)=\frac{1}{3},\ \mathrm{X}(4)=\mathrm{Y}(4)=\frac{1}{4},$$
$$\mathrm{X}(5)=\mathrm{Y}(5)=\frac{1}{5},\ \mathrm{X}(6)=\mathrm{Y}(6)=\frac{1}{6},\ \mathrm{X}(7)=\mathrm{Y}(7)=\frac{1}{7},\ \mathrm{X}(8)=\mathrm{Y}(8)=\frac{1}{8},$$
$$\mathrm{X}(9)=\mathrm{Y}(9)=\frac{1}{9},\ \mathrm{X}(10)=\mathrm{Y}(10)=\frac{1}{10},\ \mathrm{X}(11)=\mathrm{Y}(11)=\frac{1}{11},\ \mathrm{X}(12)=\mathrm{Y}(12)=\frac{1}{12}.$$

Hence, the series solution reads

$$y(\mathrm{t})=\mathrm{x(t)}=\mathrm{t}+\frac{t^2}{2}+\frac{t^3}{3}+\frac{t^4}{4}+\frac{t^5}{5}+\frac{t^6}{6}+\frac{t^7}{7}+\frac{t^8}{8}+\frac{t^9}{9}+\frac{t^{10}}{10}+\frac{t^{11}}{11}+\frac{t^{12}}{12},\ t\in[0,1.25].$$

On the other hand, by applying the MsDTM to Equations 4.1 and 4.2 with same initial conditions, it follows

$$\mathrm{X}_i(\mathrm{k}+1)=\frac{1}{k+1}\left(\mathrm{e}^{Y_i(0)}\sum_{r=0}^{k}\sum_{m=0}^{r}\frac{(-1)^m Y_i^m(0)}{(\mathrm{r}-\mathrm{m})!\mathrm{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_i}\right)$$

$$\mathrm{Y}_i(\mathrm{k}+1)=\frac{1}{k+1}\left(\mathrm{e}^{\mathrm{X}_i(0)}\sum_{r=0}^{k}\sum_{m=0}^{r}\frac{(-1)^m X_i^{\,m}(0)}{(\mathrm{r}-\mathrm{m})!\mathrm{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(\mathrm{x(t)}\right)^{r-m}\right]_{t=t_i}\right),$$

$$X_0(0)=0,\mathrm{X}_i(0)=\mathrm{x}_{i-1}(\mathrm{t}_i),\quad \mathrm{Y}_0(0)=0,\ Y_i(0)=\mathrm{y}_{i-1}(\mathrm{t}_i),\ \mathrm{i}=1,2,3,4,5.$$

Thus, we obtain the series solution

$$y(\mathrm{t})=\mathrm{x(t)}=\begin{cases}\mathrm{t}+0.5\,\mathrm{t}^2+0.3333\,\mathrm{t}^3+0.25\,\mathrm{t}^4+0.2\,\mathrm{t}^5+0.16667\,\mathrm{t}^6+0.14286\,\mathrm{t}^7\\ +0.125t^8+0.1111t^9+0.1t^{10}+0.0909t^{11}+0.0833t^{12},\quad t\in[0,0.25]\\ 0.286682+1.33333(\mathrm{t}-0.25)+0.888889(\mathrm{t}-0.25)^2+0.790124(\mathrm{t}-0.25)^3\\ +0.790124(\mathrm{t}-0.25)^4+0.842798(\mathrm{t}-0.25)^5+0.936443(\mathrm{t}-0.25)^6\\ +1.07022(\mathrm{t}-0.25)^7+1.24859(\mathrm{t}-0.25)^8+1.47981(\mathrm{t}-0.25)^9\\ +1.77577(\mathrm{t}-0.25)^{10}+2.15245(\mathrm{t}-0.25)^{11}+2.63078(\mathrm{t}-0.25)^{12},\ t\in[0.25,0.5]\\ 0.693147+2(\mathrm{t}-0.5)+2(\mathrm{t}-0.5)^2+2.66667(\mathrm{t}-0.5)^3+4(\mathrm{t}-0.5)^4+6.4(\mathrm{t}-0.5)^5\\ +10.66667(\mathrm{t}-0.5)^6+18.2857(\mathrm{t}-0.5)^7+32(\mathrm{t}-0.5)^8+56.8889(\mathrm{t}-0.5)^9\\ +102.4(\mathrm{t}-0.5)^{10}+186.182(\mathrm{t}-0.5)^{11}+341.333(\mathrm{t}-0.5)^{12},\ t\in[0.5,0.75]\\ 1.38628+3.99993(\mathrm{t}-0.75)+7.99972(\mathrm{t}-0.75)^2+21.3322(\mathrm{t}-0.75)^3\\ +63.9955(\mathrm{t}-0.75)^4+204.782(\mathrm{t}-0.75)^5+682.595(\mathrm{t}-0.75)^6+2340.28(\mathrm{t}-0.75)^7\\ 8190.85(\mathrm{t}-0.75)^8+29122.5(\mathrm{t}-0.75)^9+104839(\mathrm{t}-0.75)^{10}+381227(\mathrm{t}-0.75)^{11}\\ +1.39781\times10^6(\mathrm{t}-0.75)^{12},\ t\in[0.75,1]\\ 4.05773+57.843(\mathrm{t}-1)+1672.91(\mathrm{t}-1)^2+64510.45(\mathrm{t}-1)^3+2.79861\times10^6(\mathrm{t}-1)^4\\ +1.56645\times10^{13}(\mathrm{t}-1)^8+8.05403\times10^{14}(\mathrm{t}-1)^9+4.19282\times10^{16}(\mathrm{t}-1)^{10}\\ +2.20478\times10^{18}(\mathrm{t}-1)^{11}+1.16903\times10^{20}(\mathrm{t}-1)^{12},\ t\in[1,1.25].\end{cases}$$

This problem with the initial conditions $x(0)=0,\ \mathrm{y}(0)=0$ can be solved analytically by the phase-plane method to obtain the analytical solution $y(\mathrm{x})=\mathrm{x}$. As seen in Figure 1, the approximate series solutions calculated by the DTM and the MsDTM are the same as the analytical solution and they have the same direction with the flow of the vector fields. Moreover, the DTM and the MsDTM gave data results similar to the analytical results (Table 3).

However, if we consider the problem with the initial conditions of $x(0)=-2,\ \mathrm{y}(0)=1.$, the analytical solution obtained is $y(\mathrm{x})=\ln\left(e^x+e-e^{-2}\right)$. The data values of the approximate solutions of the DTM and the MsDTM were compared

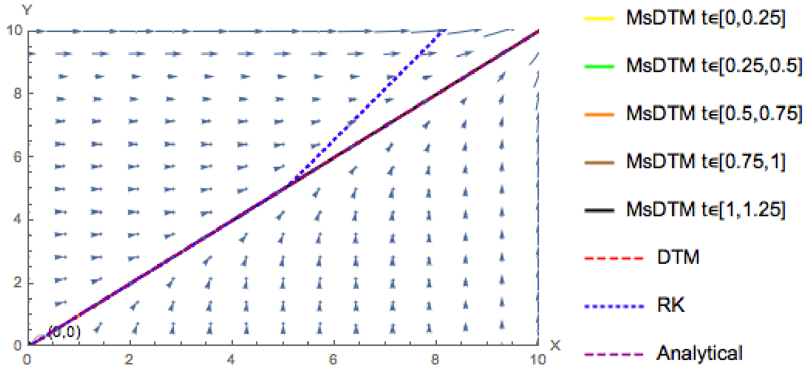

Figure 1. MsDTM: The DTM and numerical solution compared with vector field flow directions.

Table 3. DTM and MsDTM values compared with the analytical solutions.

t	x(t)	MsDTM	Analytical	Error
0.2	0.2231436	0.2231436	0.2231436	0
0.4	0.5108257	0.5108257	0.5108257	0
0.6	0.9162908	0.9162908	0.9162908	0
0.8	1.6094160	1.6094160	1.6094160	0
t	x(t)	DTM	Analytical	Error
0.2	0.2231436	0.2231436	0.2231436	0
0.4	0.5108248	0.5108248	0.5108248	0
0.6	0.9160622	0.9160622	0.9160622	0
0.8	1.5924103	1.5924103	1.5924103	0

with analytical solution and are shown in Table 4. We can see that the MsDTM results are much more similar to the analytical results than the DTM results.

The following two examples show that the proposed new transformed functions of the product of composite functions can be applied effectively to the nonlinear plane autonomous system when the analytical solutions are unavailable.

Example 4.2. Let us consider the following system of nonlinear plane autonomous

$$x' = x^2 e^y \tag{4.3}$$

$$y' = ye^x - y, \text{ for } t \in [0, 0.2], \tag{4.4}$$

subject to the initial conditions $x(0)=1,\ \mathrm{y}(0)=1.$

Applying the DTM to Equations 4.3 and 4.4 and with the initial conditions $x(0)=1, \mathrm{y}(0)=1,$ it follows that

$$X(\mathrm{k}+1) = \frac{1}{k+1}\sum_{r=0}^{k} F(\mathrm{r}) \sum_{l=0}^{k-r} X(l)\,\mathrm{X}(\mathrm{k}-\mathrm{r}-l)$$

$$Y(\mathrm{k}+1) = \frac{1}{k+1}\left(\sum_{r=0}^{k} Y(\mathrm{k}-\mathrm{r})\,\mathrm{G}(\mathrm{r}) - \mathrm{Y}(\mathrm{k})\right),$$

and the initial condition becomes $X(0)=1,\ \mathrm{Y}(0)=1,$

where

$$F(\mathrm{r}) = \mathrm{e}^{Y(0)} \sum_{l=0}^{r}\sum_{m=0}^{l} \frac{(-1)^m Y^m(0)}{(\mathrm{l}-\mathrm{m})!\mathrm{m}!}\left[\frac{1}{\mathrm{r}!}\frac{d^r}{dt^r}\big(y(\mathrm{t})\big)^{l-m}\right]_{t=0},$$

$$\mathrm{G}(\mathrm{r}) = \mathrm{e}^{\mathrm{X}(0)} \sum_{l=0}^{r}\sum_{m=0}^{l} \frac{(-1)^m X^m(0)}{(\mathrm{l}-\mathrm{m})!\mathrm{m}!}\left[\frac{1}{\mathrm{r}!}\frac{d^r}{dt^r}\big(\mathrm{x}(\mathrm{t})\big)^{l-m}\right]_{t=0}.$$

Table 4. DTM and MsDTM values compared with the analytical solutions.

t	x(t)	MsDTM	Analytical	Error
0.2	-1.4473324	1.0360783	1.0360783	0
0.4	-0.8671829	1.0996385	1.0996384	1×10^{-7}
0.6	-0.2340544	1.2161777	1.2161776	1×10^{-7}
0.8	0.5147103	1.4483531	1.4483533	2×10^{-7}
t	x(t)	DTM	Analytical	Error
0.2	-1.4473324	1.0360783	1.0360783	0
0.4	-0.8671831	1.0996383	1.0996384	2×10^{-7}
0.6	-0.2340822	1.2161499	1.2161711	2.12×10^{-5}
0.8	0.5130110	1.4466537	1.4476856	1.0319×10^{-3}

Hence, we obtain the series solution by the DTM

$$x(t)=1+2.71828\,t+9.72444t^2+38.8048t^3+164.329t^4+722.872t^5+3265.98t^6+15052.5t^7$$
$$+7045.9t^8+333873t^7+1.59826\times10^6t^{10}+7.71576\times10^6t^{11}+3.75161\times10^7t^{12},\ t\in[0,0.2]$$

$$y(t)=1+1.71828\,t+5.17077\,t^2+19.3526t^3+80.1435t^4+351.093t^5+1593.47t^6+7409.3t^7$$
$$+35062.4t^8+168146t^9+814830t^{10}+3.98192\times10^6t^{11}+1.95937\times10^7t^{12},\quad t\in[0,0.2].$$

On the other hand, by applying the MsDTM to Equations 4.3 and 4.4 we obtain

$$X_i(k+1)=\frac{1}{k+1}\sum_{r=0}^{k}F_i(r)\sum_{l=0}^{k-r}X_i(l)\,X_i(k-r-l)$$

$$Y_i(k+1)=\frac{1}{k+1}\left(\sum_{r=0}^{k}Y_i(k-r)\,G_i(r)-Y_i(k)\right),\quad X_0(0)=1,\ X_i(0)=x_{i-1}(t_i),\ Y_0(0)=1,\ Y_i(0)=y_{i-1}(t_i),\ i=1,2,3,4$$

where

$$F_i(r)=e^{Y_i(0)}\sum_{l=0}^{r}\sum_{m=0}^{l}\frac{(-1)^mY_i^m(0)}{(l-m)!m!}\left[\frac{1}{r!}\frac{d^r}{dt^r}\left(y(t)\right)^{l-m}\right]_{t=t_i},\quad G(r)=e^{X_i(0)}\sum_{l=0}^{r}\sum_{m=0}^{l}\frac{(-1)^mX_i^{\,m}(0)}{(l-m)!m!}\left[\frac{1}{r!}\frac{d^r}{dt^r}\left(x(t)\right)^{l-m}\right]_{t=t_i}.$$

The following approximate series solution is the result.

$$x(t)=\begin{cases}1+2.71828t+9.72444t^2+38.8048t^3+164.329t^4+722.872t^5+3265.98t^6+15052.5t^7\\ +70454.9t^8+333873t^9+1.59826\times10^6t^{10}+7.71576\times10^6t^{11}+3.75176\times10^7t^{12},\ t\in[0,0.05]\\ 1.1664+4.09489(t-0.05)+19.3628(t-0.05)^2+103.094(t-0.05)^3\\ +585.693(t-0.05)^4+3468.59(t-0.05)^5+21148.6(t-0.05)^6+131763(t-0.05)^7\\ +834742(t-0.06)^8+5.35899\times10^6(t-0.06)^9+3.47789\times10^7(t-0.06)^{10}\\ +2.27748\times10^8(t-0.06)^{11}+1.50275\times10^9(t-0.06)^{12},\quad t\in[0.05,0.1]\\ 1.43766+7.26784(t-0.1)+51.4129(t-0.1)^2+416.14(t-0.12)^3\\ +3626.29(t-0.1)^4+33122.1(t-0.1)^5+312581(t-0.12)^6+3.02155\times10^6(t-0.1)^7\\ +2.9749\times10^7(t-0.1)^8+2.97167\times10^8(t-0.1)^9+3.00338\times10^9(t-0.1)^{10}\\ +3.06482\times10^{10}(t-0.1)^{11}+3.15286\times10^{11}(t-0.1)^{12},\ t\in[0.1,0.15]\\ 2.02412+19.7164(t-0.15)+293.806(t-0.15)^2+5196.51(t-0.15)^3+100769(t-0.15)^4\\ +2.06843\times10^6(t-0.15)^5+4.41157\times10^7(t-0.15)^6+9.67037\times10^8(t-0.15)^7\\ +2.16368\times10^{10}(t-0.15)^8+4.91847\times10^{11}(t-0.15)^9+1.13227\times10^{13}(t-0.15)^{10}\\ +2.63346\times10^{14}(t-0.15)^{11}+6.17734\times10^{15}(t-0.15)^{12},\ t\in[0.15,0.2].\end{cases}$$

$$y(t)=\begin{cases}1+1.71828t+5.17077t^2+19.3536t^4+80.1435t^5+351.093t^6+7409.3t^7\\ +35062.4t^8+168146t^9+814830t^{10}+3.98192\times10^6t^{11}+1.95937\times10^7t^{12},\ t\in[0,0.05]\\ 1.1019+2.43565(t-0.05)+9.93481(t-0.05)^2+50.7118(t-0.05)^3\\ +286.334(t-0.05)^4+1708.87(t-0.05)^5+10559.3(t-0.05)^6+66814.5(t-0.05)^7\\ +430122(t-0.05)^8+2.8059\times10^6(t-0.05)^9+1.84864\times10^7(t-0.05)^{10}\\ +1.2283\times10^8(t-0.05)^{11}+8.21708\times10^8(t-0.05)^{12},\ t\in[0.05,0.1]\\ 1.25743+4.03738(t-0.1)+25.7226(t-0.1)^2+206.07(t-0.1)^3\\ +1823.14(t-0.1)^4+17024.4(t-0.1)^5+164483(t-0.1)^6+1.62549\times10^6(t-0.1)^7\\ +1.63413\times10^7(t-0.1)^8+1.66403\times10^8(t-0.1)^9+1.71169\times10^9(t-0.11)^{10}\\ +1.75515\times10^{10}(t-0.1)^{11}+1.85343\times10^{11}(t-0.1)^{12},\ t\in[0.1,0.15]\\ 1.57118+10.3218(t-0.15)+151.148(t-0.15)^2+2.779.76(t-0.15)^3+56212.2(t-0.15)^4\\ +1.19708\times10^6(t-0.15)^5+2.63404\times10^7(t-0.15)^6+5.92862\times10^8(t-0.15)^7\\ +1.35679\times10^{10}(t-0.15)^8+3.145\times10^{11}(t-0.15)^9+7.36425\times10^{12}(t-0.15)^{10}\\ +1.73866\times10^{14}(t-0.15)^{11}+4.13307\times10^{15}(t-0.15)^{12},\ t\in[0.15,0.2].\end{cases}$$

The approximate series solution obtained by the MsDTM and the DTM are compared graphically with the flow direction of the vector fields (Figure 2). We can see that the MsDTM result is in better agreement with vector field than the DTM result.

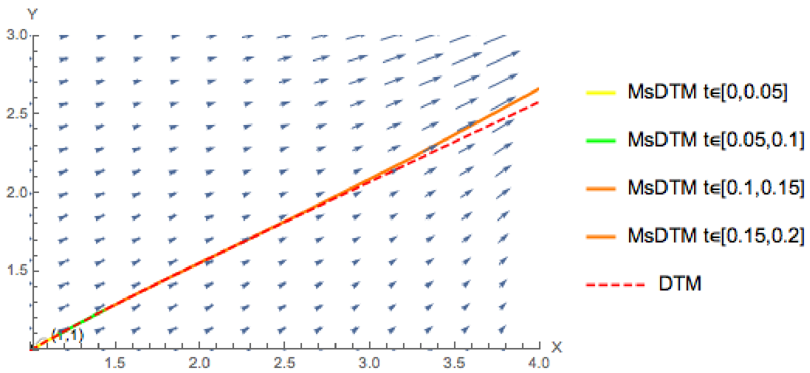

Figure 2. MsDTM and DTM compared with vector field flow directions.

Example 4.3. Let us consider the following system of nonlinear plane autonomous

$$x'=2x+\sin y \tag{4.5}$$

$$y'=x(y^2+1),\ \text{for}\ t\in[0,0.4], \tag{4.6}$$

subject to the initial condition $x(0)=1,\ y(0)=1.$

Applying the DTM of Equations 4.5 and 4.6, we obtain

$$X(k+1)=\frac{1}{k+1}\left(2X(k)+\sum_{r=0}^{k}\frac{d^r}{dt^r}\sin(t)\Big|_{t=Y(0)}\sum_{m=0}^{r}\frac{(-1)^mY^m(0)}{(r-m)!m!}\left[\frac{1}{k!}\frac{d^k}{dt^k}(y(t))^{r-m}\right]_{t=0}\right)$$

$$Y(k+1)=\frac{1}{k+1}\left(X(k)+\sum_{r=0}^{k}\sum_{l=0}^{r}Y(l)Y(r-l)X(k-r)\right),$$

and the initial condition becomes $x(0)=1,\ y(0)=1.$

Then, we obtain the series solution

$$x(\mathrm{t}) = 1 + 2.84147\,\mathrm{t} + 3.38177\,\mathrm{t}^2 + 2.56549t^3 + 0.498022t^4 - 3.62605t^5 - 13.1978t^6$$
$$-36.6427t^7 - 93.1066t^8 - 224.685t^9 - 520.598t^{10} - 1160.66t^{11} - 2481.54t^{12},\ t \in [0, 0.4]$$

$$y(\mathrm{t}) = 1 + 2\,\mathrm{t} + 4.84147\,\mathrm{t}^2 + 10.6041t^3 + 24.528t^4 + 57.5466t^5 + 134.91t^6$$
$$+315.164t^7 + 733.76t^8 + 1702.66t^9 + 3938.03t^{10} + 9079.66t^{11} + 20873.9t^{12},\ t \in [0, 0.4].$$

On the other hand, by applying the MsDTM to Equationa 4.5 and 4.6, it follows

$$X_i(\mathrm{k}+1) = \frac{1}{k+1}\left(2X_i(\mathrm{k}) + \sum_{r=0}^{k} \frac{d^r}{dt^r}\sin(\mathrm{t})\Big|_{t=Y_i(0)} \sum_{m=0}^{r} \frac{(-1)^m Y_i^m(0)}{(\mathrm{r}-\mathrm{m})!\mathrm{m}!} \left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m} \right]_{t=t_i} \right)$$

$$Y_i(\mathrm{k}+1) = \frac{1}{k+1}\left(X_i(\mathrm{k}) + \sum_{r=0}^{k}\sum_{l=0}^{r} Y_i(l)\,\mathrm{Y}_i(\mathrm{r}-l) X_i(\mathrm{k}-\mathrm{r}) \right),$$
$$X_0(0) = 1,\ \mathrm{X}_i(0) = \mathrm{x}_{i-1}(\mathrm{t}_i),\ \mathrm{Y}_0(0) = 1,\ Y_i(0) = y_{i-1}(\mathrm{t}_i),\ \mathrm{i} = 1, 2, 3, 4, 5.$$

Hence, we obtain the series solution

$$x(\mathrm{t}) = \begin{cases}
1 + 2.84147\,\mathrm{t} + 3.38177\,\mathrm{t}^2 + 2.56549t^3 + 0.498022t^4 - 3.62605t^5 - 13.1978t^6 \\
-36.6427t^7 - 93.1066t^8 - 224.685t^9 - 520.598t^{10} - 1160.66t^{11} - 2481.54t^{12},\ t \in [0, 0.08] \\
1.25028 + 3.43174(\mathrm{t}-0.08) + 3.98652(\mathrm{t}-0.08)^2 + 2.28135(\mathrm{t}-0.08)^3 \\
-3.27511(\mathrm{t}-0.08)^4 - 19.3406(\mathrm{t}-0.08)^5 - 67.1434(\mathrm{t}-0.08)^6 \\
-205.262(\mathrm{t}-0.08)^7 - 584.413(\mathrm{t}-0.08)^8 - 1563.52(\mathrm{t}-0.08)^9 \\
-3898.43(\mathrm{t}-0.08)^{10} - 8808.52(\mathrm{t}-0.08)^{11} - 16669.2(\mathrm{t}-0.08)^{12},\ t \in [0.08, 0.16] \\
1.55128 + 4.10089(\mathrm{t}-0.16) + 4.24803(\mathrm{t}-0.16)^2 - 1.14073(\mathrm{t}-0.16)^3 \\
-23.7465(\mathrm{t}-0.16)^4 - 108.548(\mathrm{t}-0.16)^5 - 404.828(\mathrm{t}-0.16)^6 \\
-1336.03(\mathrm{t}-0.16)^7 - 3807.58(\mathrm{t}-0.16)^8 - 8159.94(\mathrm{t}-0.16)^9 \\
-2936.53(\mathrm{t}-0.16)^{10} + 104910(\mathrm{t}-0.16)^{11} + 850945(\mathrm{t}-0.16)^{12},\ t \in [0.16, 0.24] \\
1.90448 + 4.67667(\mathrm{t}-0.24) + 2.1336(\mathrm{t}-0.24)^2 - 22.5505(\mathrm{t}-0.24)^3 \\
-142.022(\mathrm{t}-0.24)^4 - 585.377(\mathrm{t}-0.24)^5 - 1394.54(\mathrm{t}-0.24)^6 + 4460.95(\mathrm{t}-0.24)^7 \\
+93830.2(\mathrm{t}-0.24)^8 + 865886(\mathrm{t}-0.24)^9 + 6.29241\times10^6(\mathrm{t}-0.24)^{10} \\
+3.98717\times10^7(\mathrm{t}-0.24)^{11} + 2.27165\times10^8(\mathrm{t}-0.24)^{12},\ t \in [0.24, 0.32] \\
2.27311 + 420097(\mathrm{t}-0.32) - 9.88838(\mathrm{t}-0.32)^2 - 38.0005(\mathrm{t}-0.32)^3 + 1179.65(\mathrm{t}-0.32)^4 \\
+24086.5(\mathrm{t}-0.32)^5 + 281349(\mathrm{t}-0.32)^6 + 2.24486\times10^6(\mathrm{t}-0.32)^7 \\
+8.20097\times10^6(\mathrm{t}-0.32)^8 - 1.15647\times10^8(\mathrm{t}-0.32)^9 - 3.29081\times10^9(\mathrm{t}-0.32)^{10} \\
-5.19089\times10^{10}(\mathrm{t}-0.32)^{11} - 6.53363\times10^{11}(\mathrm{t}-0.32)^{12},\ t \in [0.32, 0.4].
\end{cases}$$

$$y(\mathrm{t}) = \begin{cases}
1 + 2\,\mathrm{t} + 4.84147\,\mathrm{t}^2 + 10.6041t^3 + 24.528t^4 + 57.5466t^5 + 134.91t^6 + 315.164t^7 \\
+733.76t^8 + 1702.66t^9 + 3938.03t^{10} + 9079.66t^{11} + 20873.9t^{12},\ t \in [0, 0.08] \\
1.19765 + 3.04364(\mathrm{t}-0.08) + 8.7346(\mathrm{t}-0.08)^2 + 24.1547(\mathrm{t}-0.08)^3 \\
+69.2557(\mathrm{t}-0.08)^4 + 199.329(\mathrm{t}-0.08)^5 + 571.305(\mathrm{t}-0.08)^6 \\
+1629.29(\mathrm{t}-0.08)^7 + 4624.49(\mathrm{t}-0.08)^8 + 13067.8(\mathrm{t}-0.08)^9 \\
+36779.9(\mathrm{t}-0.08)^{10} + 103170(\mathrm{t}-0.08)^{11} + 288632(\mathrm{t}-0.08)^{12},\ t \in [0.08, 0.16] \\
1.5131 + 5.10287(\mathrm{t}-0.16) + 18.7225(\mathrm{t}-0.16)^2 + 68.529(\mathrm{t}-0.16)^3 \\
+254.775(\mathrm{t}-0.16)^4 + 942.871(\mathrm{t}-0.16)^5 + 3463.75(\mathrm{t}-0.16)^6 + 12638.4(\mathrm{t}-0.16)^7 \\
+45855(\mathrm{t}-0.16)^8 + 165680(\mathrm{t}-0.16)^9 + 597192(\mathrm{t}-0.16)^{10} \\
+2.15185\times10^6(\mathrm{t}-0.16)^{11} + 7.76857\times10^6(\mathrm{t}-0.16)^{12},\ t \in [0.16, 0.24]
\end{cases}$$

$$y(t)=\begin{cases} 2.09104+10.2318(t-0.24)+53.3091(t-0.24)^2+278.516(t-0.24)^3 \\ +1449.56(t-0.24)^4+7466.42(t-0.24)^5+38168.7(t-0.24)^6+194558(t-0.24)^7 \\ +994211(t-0.24)^8+5.12151\times10^6(t-0.24)^9+2.67247\times10^7(t-0.24)^{10} \\ +1.41713\times10^8(t-0.24)^{11}+7.64272\times10^8(t-0.24)^{12},\ t\in[0.24,0.32] \\ 3.4941+30.0249(t-0.32)+266.217(t-0.32)^2+2342.96(t-0.32)^3+20645.6(t-0.32)^4 \\ +185021(t-0.32)^5+1.70474\times10^6(t-0.32)^6+1.61372\times10^7(t-0.32)^7 \\ +1.55164\times10^8(t-0.32)^8+1.49064\times10^9(t-0.32)^9+1.40754\times10^{10}(t-0.32)^{10} \\ +1.28807\times10^{11}(t-0.32)^{11}+1.12816\times10^{12}(t-0.32)^{12},\ t\in[0.32,0.4]. \end{cases}$$

Similar to the previous examples, the MsDTM result is in better agreement with the flow of the vector field than the DTM result (Figure 3).

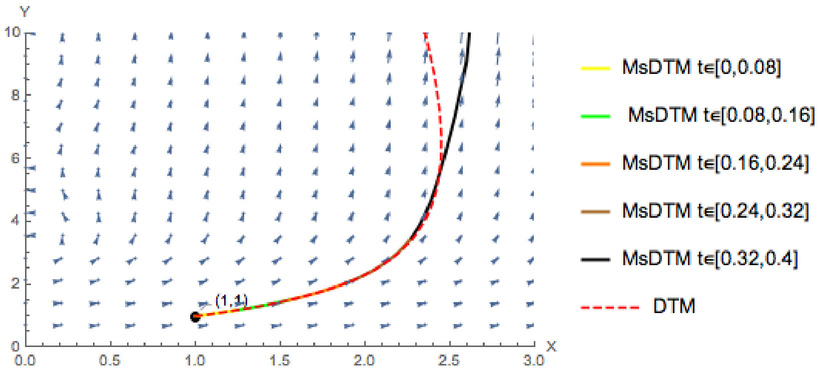

Figure 3. MsDTM and DTM compared with vector field flow directions.

5. Conclusions

The MsDTM combined with our new formulae have been successfully applied to solve nonlinear plane autonomous systems. Three different examples were solved and the series solutions of the DTM and the MsDTM were obtained. These are compared with the analytical solutions calculated by the phase-plane method in the first example and compared to the vector fields flow directions in the second and the third examples. The results of the MsDTM were more similar to the analytical solution and to the vector field flow direction than the DTM results. Therefore, this method based on our new transformed functions is a reliable and efficient mathematical tool for solving nonlinear plane autonomous systems.

Acknowledgements

This paper has been supported by scholarships for graduate student from the Faculty of Science, KMITL.

References

Chang, S. H., & Chang, I. L. (2008). A new algorithm for calculating the one-dimensional differential transform of nonlinear functions. *Applied Mathematics and Computation, 195*, 799-808.

Damirchi, J., & Shamami, T. R. (2016). Differential transform method for a nonlinear system of differential equations arising in HIV infection of *CD4⁺T* cells. *International Journal of Nonlinear Analysis and Applications, 7*(2), 269-277.

Ebenezer, B., Freihet, A., Khan, M. A., & Khan, A. (2016). Application of multi-step differential transform method to solve system of nonlinear fractional differential algebraic equations. *Journal of Applied Environmental and Biological Sciences, 6*(5), 83-95.

Ertürk, A. S., Odibat, Z. M., & Momami, S. (2012). The multi-step differential transform method and its application to determine the solutions of non-linear oscillators. *Advances in Applied Mathematics and Mechanics, 4*(4), 422-438.

Fatoorehchi, H., & Abolhasemi, H. (2013). Improving the differential transform method: A novel technique to obtain the differential transforms of nonlinearities by the Adomain polynomials. *Applied Mathematical Modeling, 37*(8), 6008-6017.

Fatoorehchi, H., & Abolhasemi, H. (2014). An explicit analytic solution to the Thomas-Fermi equation by the improved differential transform method. *Acta Physica Polonica A, 125*(5), 1083-1087.

Mahgoub, A. A. A., & Alshikh, A. A. (2017). Application of the differential transform method for the nonlinear differential equations. *American Journal of Applied Mathematics, 5*(1), 14-18.

Methi, G. (2016). Solution of differential equations using differential transform method. *Asian Journal of Mathematics and Statistics, 9*, 1-5.

Mirzaee, F. (2011). Differential transform method for solving linear and nonlinear systems of ordinary differential equations. *Applied Mathematics Sciences, 70,* 3465-3472.

Moon, S. D., Bhosale, A. B., Gajbhiye, P. P., & Lonare, G. G. (2014). Solution of nonlinear differential equations y using differential transform method. *International Journal of Mathematics and Statistics Invention, 2*(3), 78-82.

Odibat, Z. M., Bertelle, C., Aziz-Alaoui, M. A., & Duchamp, G. H. E. (2010). A multi-step differential transform method and application to non-chaotic systems. *Compute and Mathematics with Applications, 59*, 1462-1472.

Patil, N., & Khambayat, A. (2014). Differential transform method for system of linear differential equations. *Research Journal of Mathematical and Statistical Sciences, 2*(3), 4-6.

Rashidi, M. M., Chamkha, A. J., & Keimanesh, M. (2011). Application of multi-step differential transform method on flow of a second-grade fluid over a stretching or shrinking sheet. *American Journal of Computational Mathematics, 6*, 119-128.

Slevinsky, R. M., & Safouhi, H. (2009). New formulae for higher order derivatives and applications. *Journal of Computational and Applied Mathematics, 233*, 405-419.

Weisstein, E. W. (n.d.). General identities: Differentiation. Retrieved from Math Word-A Wolfram Web Research, http://functions.wolfram.com/GeneralIdentities/9/.

Zhou, J. K. (1986). *Differential transformation and its applications for electrical circuits*. Wuhan, China: Huazhong University Press.

Zill, D. G., & Wright, W. S. (2014). *Advanced engineering mathematics* (5th ed.). Burlington, MA: Jones and Bartlett.

Zurigat, M., & Ababneh, M. (2015). Application of the multi-step differential transform method to solve a fractional human T-cell lymphotropic virus I (HTLV-I) infection of $CD4^{+}T$-cells. *Journal of Mathematics and Applications, 38*, 171-181.