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Original Article

Flexible Lomax distribution

Muhammad Ijaz*, Muhammad Asim, and Alamgir Khalil

Department of Statistics, University of Peshawar, Peshawar, Khyber Pakhtunkhwa, 25130 Pakistan

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Abstract

In this paper, a new probability model is produced, which is actually a modification of the Lomax distribution, called Flexible Lomax (FL). The Flexible Lomax performs better than Lomax and its other invariants. The statistical properties of the Flexible Lomax distribution including quantile function, maximum likelihood estimation, order statistics, and r^{th} moments are derived. To illustrate the efficiency of the proposed distribution, we use two real-life data sets and the results are then compared by means of AIC, CAIC, BIC, and HQIC. It is shown that the proposed distribution fits these data better than Lomax and its different variants.

Keywords: R^{th} moment, quantile function, characteristic function, median, mode, order statistics, TTT plots, maximum likelihood estimation

1. Introduction

Statistician often need various lifetime probability distributions like Lomax, Weibull-Lomax (Tahir, Cordeiro, Mansoor, Zubair, & Tahir, 2015), and Gamma-Lomax (Cordeiro, Ortega, & Popovic, 2015). These distributions have desirable properties and physical interpretations. This paper present a new modification of the Lomax distribution, called Flexible Lomax (FL).

Prior literature provides different modifications of the Lomax distribution. Ghitany, Awadhi, and Alkhalfan (2007) presented *Marshall–Olkin extended Lomax* distribution and its application to censored data. The probability density function of $MOEL(\alpha\beta\gamma)$ is

*Corresponding author
Email address: ijaz.statistics@gmail.com

$$f(x)=\alpha\beta\gamma\frac{(1+\beta x)^{\gamma-1}}{\left[\left((1+\beta x)^{\gamma}-\bar{\alpha}\right)\right]^{2}},\quad \gamma>1$$

Lemonte and Cordeiro (2013) introduced an *extended Lomax* distribution with parameters $(\alpha,\ \beta,\ a,\ \eta,\ c)$ and the probability density function

$$f(x)=\frac{c\alpha\beta^{\alpha}(\beta+x)^{-(\alpha+1)}}{B\left(ac^{-1},\eta+1\right)}\left\{1-\left(\frac{\beta}{x+\beta}\right)^{\alpha}\right\}^{a-1}\left[1-\left\{1-\left(\frac{\beta}{x+\beta}\right)^{\alpha}\right\}^{c}\right]^{\eta} \quad (1.1)$$

For other modifications of the Lomax distribution, we refer to Afify, Nofal, Yousof, El Gebaly, and (2015), Ashour and Eltehiwy (2013, 1(6), 7(7)), Al-Zahrani and Sagor (2014), El-Bassiouny, Abdo and Shahen (2015) and Shams (2013). Dias, Alizadeh and Cordeiro (2016) introduced *Beta Nadarajah-Haghighi* distribution. Domma and Condino

(2013) presented the *beta-Dagum* distribution. Korkmaz and Genç (2017) introduced a new generalized two-sided class of distributions with an emphasis on a two-sided generalized normal distribution. For the different families of distributions, we refer to Aldeni and Famoye (2017), Cordeiro (2017), Alzaatreh, Famoye and Ghosh (2016), Alzaatreh and Famoye (2014), and Nasir, Aljarrah, Jamal, and Tahir (2017). Otunuga (2017) has worked on the *Pareto-g extended Weibull* distribution. For further details on relevant distributions, see Eddy (2007), El-Gohary, Alshamrani and Otaibi (2013), Famoye, Alzaatreh (2013), Aarset (1987), and Boyd (1988).

2. Flexible Lomax (FL) Distribution

The main objective of the present paper is to introduce a new modification of the Lomax distribution with increased number of parameters. The proposed distribution (FL) has three parameters, and its cumulative distribution function takes the form

$$F(x)=1-\left[1+\left(\frac{x}{b}\right)^{c}\right]^{-a}, \qquad x>0 \tag{2.1}$$

Other statistical characteristics implied by (2.1) are as follows.

$$f(x)=\frac{ac}{b^{c}}x^{c-1}\left[1+\left(\frac{x}{b}\right)^{c}\right]^{-(a+1)}, \qquad x>0 \tag{2.2}$$

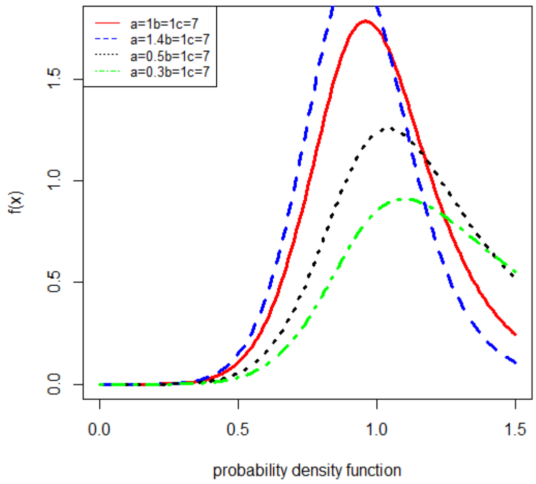
The hazard rate function of the FL distribution is

$$h(x)=\frac{ac}{b^{c}}\frac{x^{c-1}}{1+\left(\frac{x}{b}\right)^{c}} \tag{2.3}$$

The survival function of the FL distribution is

$$S(x)=\left[1+\left(\frac{x}{b}\right)^{c}\right]^{-a} \tag{2.4}$$

Figure 1 shows the graphs of the probability density function and the cumulative distribution function, for various parameter values.



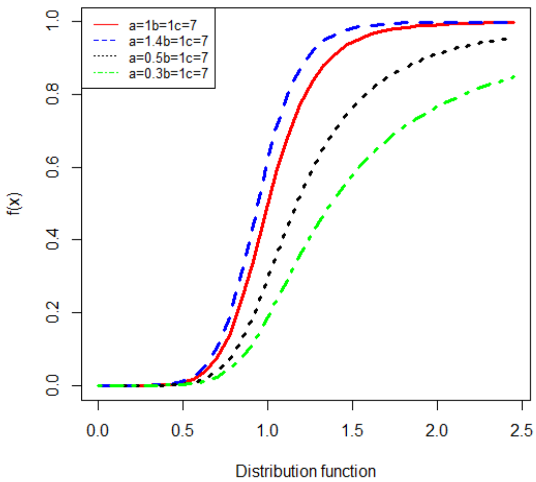


Figure 1. The Pdf and Cdf of Flexible Lomax.

3. Behavior of the Pdf and Hazard Rate Function

Theorem 3.1. The Pdf of *Flexible Lomax* distribution $f(x)$ is

a. increasing when $a,b>0, c>1$

b. decreasing when $a>0, 0<c\leq 1, b>0$

Proof. The first derivative from Equation (2) is

$$f'(x)=\frac{d}{dx}\left(\frac{ac}{b^{c}}x^{c-1}\left[1+\left(\frac{x}{b}\right)^{c}\right]^{-(a+1)}\right)$$

$$= \frac{-\left(acx^{c-2}\left[1+\left(\frac{x}{b}\right)^{c}\right]^{-(a+2)} \right)\left((ac+1)\left(\frac{x}{b}\right)^{c} - c + 1 \right)}{b^{c}} \quad (3.1.1)$$

Since $f'(x) < 0$ for $0 < c \le 1$ the $f(x)$ is decreasing. If $c > 1$, then $f'(x) = 0$ gives the mode

$$x_m = b\left[\frac{(c-1)}{(ac+1)}\right]^{\frac{1}{c}}, \qquad c > 1 \; and \; b > 0 \quad (3.1.2)$$

For parameter values $a, b > 0, c > 1$,

$$f''(x) = \frac{acx^{c-3}\left[\left(\frac{x}{b}\right)^{c}+1\right]^{-a-3}\left[(ac+1)(ac+2)\left(\frac{x}{b}\right)^{2c} - (c-1)\left((3a+1)c+4\right)\left(\frac{x}{b}\right)^{c} + (c-1)(c-2)\right]}{b^{c}}$$

satisfies $f''(x) > 0$, and hence $f(x)$ is convex. Figures 1 shows plots for various parameter choices.

Theorem 3.2. The hazard rate function of *Flexible Lomax* (a,b,c) distribution $h(x)$ is

a. increasing when $a, b > 0, c > 1$

b. decreasing when $a > 0, 0 < c \le 1, b > 0$

Proof. The derivative of the hazard rate function in Equation (3) is given by

$$h'(x) = \frac{-acx^{c-2}\left(\left(\frac{x}{b}\right)^{c} - c + 1\right)}{b^{c}\left(\left(\frac{x}{b}\right)^{c} + 1\right)2} \quad (3.2.1)$$

$h'(x) < 0$ for $0 < c \le 1$, so the hazard rate function is decreasing. If $c > 1$, then $h'(x) = 0$ specifies the global maximum of $h(x)$ at

$$x_m = b(1-c)^{\frac{1}{c}} \quad (3.2.2)$$

Now, the second derivative of the hazard rate function is

$$h''(x) = \frac{acx^{c-3}\left(2\left(\frac{x}{b}\right)^{c} + \left(-c^2 - 3c + 4\right)\left(\frac{x}{b}\right)^{c} + c^2 - 3c + 2\right)}{b^{c}\left(\left(\frac{x}{b}\right)^{c} + 1\right)^{3}} \quad (3.2.3)$$

For parameter values $a, b > 0, c > 1$, $h''(x_m) < 0$ and hence, the hazard rate function has the ability to model shapes both monotonically and non-monotonically in bathtub shape. Figure 2 shows plots of the hazard function of the *Flexible Lomax* distribution for various choices of the parameters.

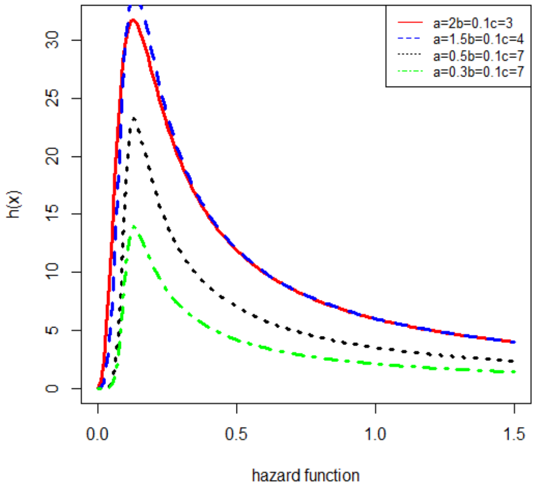


Figure 2. Hazard rate function of the Flexible Lomax

4. Quantile Function and Median

Among the related statistical functions, the quantile function is often useful. The quantile function $Q_{(FL)}(x)$ of the $FL(a,b,c)$ is the real solution to the following equation

$$1-\left[1+\left(\frac{x}{b}\right)^{c}\right]^{-a}=u \qquad (4.1)$$

where $u \sim$ Uniform (0,1).

Solving (4.1) for x, the result is

$$x=\left(e^{\frac{-\left[\log(1-u)+ac\log(b)\right]}{ac}}\right) \qquad (4.2)$$

For the median, we have to put $u=\frac{1}{2}$ in equation (4.2) to find

$$x=\left(e^{\frac{-\left[\log\left(\frac{1}{2}\right)+ac\log(b)\right]}{ac}}\right)$$

5. Rth Moments

Theorem 5.1. If X has a *Flexible Lomax* distribution with parameters a,b,c then the rth moments (about the origin) of X, say u'_r do not exist.

$$u'_r=E(x^r)=\int_0^\infty x^r f(x)dx$$

$$u'_r=\int_0^\infty\left(\frac{ac}{b^c}x^{r+c-1}\left[1+\left(\frac{x}{b}\right)^{c}\right]^{-(a+1)}\right)dx$$

$$=\frac{ac}{b^c}\int_0^\infty\left(x^{r+c-1}\left(1+\left(\frac{x}{b}\right)^{c}\right)^{-(a+1)}\right)dx \qquad (5.1.1)$$

Let, $y=1+\left(\frac{x}{b}\right)^{c}$, then (5.1.1) can be rewritten as

$$=ab^r\int_1^\infty\left((y-1)^{\frac{r}{c}}(y)^{-(a+1)}\right)dy \qquad (5.1.2)$$

Now, by using the binomial expansion we have

$$(y-1)^{\frac{r}{c}}=\sum_{k=0}^{\infty}y^k(-1)^{\frac{r}{c}+k}\binom{\frac{r}{c}}{k}$$

So, equation (5.1.2) takes the form

$$=ab^r\sum_{k=0}^{\infty}(-1)^{\frac{r}{c}+k}\binom{\frac{r}{c}}{k}\int_1^\infty(y)^{k-a-1}dy \qquad (5.1.3)$$

In (5.1.3) the integral is undefined and thus the statement proved. However, one can find the moments if $k<a$, and they take the form

$$u'_r=ab^r\sum_{k=0}^{\infty}(-1)^{\frac{r}{c}+k}\frac{\binom{\frac{r}{c}}{k}}{a-k} \qquad (5.1.4)$$

6. Order Statistics

Let $X_1, X_2, X_3 ... X_n$ be ordered random variables, then the pdf of the i^{th} order statistic is,

$$f_{(i;n)}(x) = \frac{n!}{(i-1)!(n-i)!} f(x)F(x)^{(i-1)}[1-F(x)]^{(n-i)}, \tag{6.1}$$

The 1st and nth order probability density functions of FL can be obtained by substituting (2.1) and (2.2) in (6.1) is given by,

$$f_{(1:n)}(x) = n\left(\frac{ac}{b^c} x^{c-1}\left(1+\left(\frac{x}{b}\right)^c\right)^{-(a+1)}\right)\left(1+\left(\frac{x}{b}\right)^c\right)^{-a(n-1)} \tag{6.2}$$

$$f_{(n:n)}(x) = n\left(\frac{ac}{b^c} x^{c-1}\left(1+\left(\frac{x}{b}\right)^c\right)^{-(a+1)}\right)\left(1-\left(1+\left(\frac{x}{b}\right)^c\right)^{-a}\right)^{n-1} \tag{6.3}$$

7. Parameter Estimation

In this section, the usual maximum likelihood approach is used to estimate the unknown parameters in $FL(a,b,c)$ based on complete information. Let us assume that we have a sample $X_1, X_2, X_3 ... X_n$ from $FL(a,b,c)$. The likelihood function is given by

$$L = \prod_{i=1}^{n} f\left(x_i; a,b,c\right) \tag{7.1}$$

Substituting (2.2) in (7.1), we get

$$L = \prod_{i=1}^{n}\left(\frac{ac}{b^c} x^{c-1}\left[1+\left(\frac{x}{b}\right)^c\right]^{-(a+1)}\right) \tag{7.2}$$

Then the log-likelihood function is

$$\ell = n\log\left(\frac{ac}{b^c}\right) + (c-1)\sum_{i=1}^{n} \log x_i - (a+1)\sum_{i=1}^{n}\frac{\log x_i^c}{b^c} \tag{7.3}$$

Now we have to compute the first partial derivatives of (7.3) and on setting them equal to zero we have

$$\frac{n}{a} - \sum_{i=1}^{n}\frac{\log x_i^c}{b^c} = 0 \tag{7.4}$$

$$c(a+1)\sum_{i=1}^{n}\frac{\log x_i^c}{b^{c+1}} - \frac{nc}{b} = 0 \tag{7.5}$$

$$\frac{n}{c} + \sum_{i=1}^{n}\log x_i - \frac{c^2(a+1)\sum_{i=1}^{n}\log x_i}{b^{c+1}} = 0 \tag{7.6}$$

The above equations from (7.4) to (7.6) are not in closed form, instead numerical methods are needed to get the MLE.

8. Asymptotic Confidence Bounds

Since, the MLE of the unknown parameters a,b,c are not in closed form, it is not possible to derive the exact distribution of the MLE. We have derived asymptotic confidence bounds for the unknown parameters of $FL(a,b,c)$ based on the asymptotic distribution of MLE. For the information matrix, we have to find the second partial derivatives of the equations from (7.4) to (7.6), and these are

$$\frac{\partial\ell}{\partial a^2} = I_{11} = -\frac{n}{a^2} \tag{8.1}$$

$$\frac{\partial\ell}{\partial ab} = I_{12} = c\sum_{i=1}^{n}\frac{\log x_i^c}{b^{c+1}} \tag{8.2}$$

$$\frac{\partial\ell}{\partial ac} = I_{13} = -c\sum_{i=1}^{n}\frac{\log x_i^{\,c}}{b^{c+1}} \tag{8.3}$$

$$\frac{\partial\ell}{\partial b^2} = I_{22} = \frac{nc}{b^2} - \frac{2c(c+1)(a+1)\sum_{i=1}^{n}\log x_i}{b^{c+2}} \tag{8.4}$$

$$\frac{\partial \ell}{\partial bc}=I_{23}=-c(a+1)\left(\frac{l\log b-2}{b^{c+1}}\right)\sum_{i=1}^{n} l\log x_i-\frac{n}{b} \tag{8.5}$$

$$\frac{\partial \ell}{\partial c^2}=I_{33}=-\frac{n}{c^2} -c(a+1)\left(\frac{l\log b-2}{b^{c+1}}\right)\sum_{i=1}^{n} l\log x_i \tag{8.6}$$

The information matrix is

$$I=-\begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}$$

Hence the variance-covariance matrix is approximated as

$$V=\begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix}=\begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}^{-1}$$

In order to obtain an estimate of V, we have to replace the parameters by the corresponding MLE's, as here

$$\hat{v}=\begin{pmatrix} \hat{I}_{11} & \hat{I}_{12} & \hat{I}_{13} \\ \hat{I}_{21} & \hat{I}_{22} & \hat{I}_{23} \\ \hat{I}_{31} & \hat{I}_{32} & \hat{I}_{33} \end{pmatrix}^{-1} \tag{8.7}$$

By using the above variance-covariance matrix, one can derive the (1 - β) 100% confidence intervals for the parameters a, b, c in the following forms

$$a \pm Z_{\frac{\beta}{2}}\sqrt{\operatorname{var}(a)},\quad \hat{b} \pm Z_{\frac{\beta}{2}}\sqrt{\operatorname{var}(\hat{b})},\quad \hat{c} \pm Z_{\frac{\beta}{2}}\sqrt{\operatorname{var}(\hat{c})}$$

Here $Z_{\frac{\beta}{2}}$ is the upper $\left(\frac{\beta}{2}\right)^{th}$ percentile of the standard normal distribution.

9. Shannon Entropy

Theorem 9.1. If a random variable X has $FL(a,b,c)$ then the Shannon entropy $S_H(x)$ is

$$S_H(x)=-\int_0^\infty f(x)logf(x)dx$$

$$=-\int_0^\infty f(x)log\left(\frac{ac}{b^c}x^{c-1}\left[1+\left(\frac{x}{b}\right)^c\right]^{-(a+1)}\right)dx$$

$$=-\left[\int_0^\infty f(x)\log\left(\frac{ac}{b^c}\right)dx+(c-1)\int_0^\infty f(x)\log(x)dx\right.$$

$$\left.+(-a-1)\int_0^\infty f(x)l\log\left(1+\left(\frac{x}{b}\right)^c\right)dx\right]$$

On solving Equation (9.1.1) the result is

$$S_H(x)=\left[1-l\log\left(\frac{ac}{b^c}\right)-c+c(a+1)(1-l\log b)\right] \tag{9.1.2}$$

10. Simulation

Simulation is a statistical tool used to check the effectiveness of a model with random data. We choose two sets of parameter values with different sample sizes. Equation (4.2) is used in the simulation study to generate data from the FL distribution. The simulation experiment is repeated 100 times each, with sample sizes n = 30, 50 and 70, and with parameters

$$(a,b,c)=(0.1171833, 1.0205799,\ 12.1908030),$$
$$(0.2171833,\ 1.0205799,\ 13.1908030),$$

to sample the distribution of the proposed model. The average bias and the mean square error (MSE) are estimated. The values in Table 1 clearly show that as the sample size increases, both the bias and the MSE decrease. So, we conclude that the sampling distribution will be approximately equal to the true distribution as we increase the sample size.

11. Applications

In this section, we provide an application of the FL distribution to two real data sets to illustrate its usefulness and to compare its goodness-of-fit with other modified Lomax distributions: that is with the *Exponential Lomax* (EL), *Weibull Lomax* (WL), *POLO* distribution, and *Lomax* distri-butions (L) using Kolmogorov–Smirnov (K–S) statistic,

Table 1. Bias and MSE for the $FL(a,b,c)$ estimators.

a,b,c	n	$MSE(a)$	$MSE(b)$	$MSE(c)$	$Bias(a)$	$Bias(b)$	$Bias(c)$
0.1171833,	30	0.001083286	0.002753764	16.33397	0.006100566	0.0163594	2.865952
1.0205799,	50	0.0007489411	0.002113883	10.90313	0.004527526	0.0187895	2.540668
12.1908030	70	0.0004189042	0.000968446	9.954652	0.009813031	0.009749671	2.517321
0.2171833,	30	0.002761192	0.001604763	20.01106	0.002984398	0.0210973	3.502663
1.0205799,	50	0.002249977	0.001255249	18.95682	0.01282321	0.0227793	3.521208
13.1908030	70	0.0009731424	0.000827839	12.50464	0.008513989	0.0198048	2.909101

Akaike information criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian information criterion (BIC), and Hannan Quinn information criterion (HQIC). The mathematical forms of these criteria are

$$AIC = -2L(\hat{\psi}; y_i) + 2p \tag{11.1}$$

$$AICc = AIC + \frac{2p(p+1)}{n-p-1} \tag{11.2}$$

$$CAIC = -2L + P\{\log(n)+1\} \tag{11.3}$$

$$BIC = P\log(n) - 2L(\hat{\psi}; y_i) \tag{11.4}$$

$$HQIC = -2L_{max} + 2P\log\{\log(n)\} \tag{11.5}$$

where L is the maximized likelihood function and y_i is the given random sample, $\hat{\psi}$ is the maximum likelihood estimator, and p is the number of parameters in the model.

Data set 1: Losses due to wind catastrophes.

The first data set represents the losses due to wind catastrophes recorded in 1977, taken from Boyd (1988). The data set consists of 40 exemplars that were recorded to the nearest $1,000,000 and include only losses of $ 2,000,000 or more. The following data set provides the losses (in millions of dollars): 2,2,2,2,2,2,2,2,2,2,2,2,3,3,3,3,4,4,4,5,5,5,5,6,6,6,6, 8,8,9,15,17,22,23,24,25,27,32,43.

The maximum likelihood estimates are given in Table 2, and Table 3 presents the values of (AIC), (CAIC), (BIC), and (HQIC).

Data set 2: Breaking stresses of carbon fibers.

The second real data set represents the breaking stresses of carbon fibers of 50 mm length presented by Nichols and Padgett (2006). The data are as follows. 3.70, 2.74,2.73,2.50,3.60,3.11,3.27,2.87,1.47,3.11,4.42,2.41,3.19,3.22,1.69,3.28,3.09,1.87,3.15,4.90,3.75,2.43,2.95,2.97,3.39,2.96,2.53,2.67,2.93,3.22,3.39,2.81,4.20,3.33,2.55,3.31,3.31,2.85,2.56,3.56,3.15,2.35,2.55,2.59,2.38,2.81,2.77,2.17,2.83,1.92,1.41,3.68,2.97,1.36,0.98,2.76,4.91,3.68,1.84,1.59,3.19,1.57,0.81,5.56,1.73,1.59,2.00,1.22,1.12,1.71,2.17,1.17,5.08,2.48,1.18,3.51,2.17,1.69,1.25,4.38,1.84,0.39,3.68,2.48,0.85,1.61,2.79,4.70,2.03,1.80,1.57,1.08,2.03,1.61,2.12,1.89,2.88,2.82,2.05,3.65.

The maximum likelihood estimates for the data on breaking stresses of the carbon fibers are given in Table 4, and Table 5 presents the values of (AIC), (CAIC), (BIC), and (HQIC).

Table 2 and Table 4 present the maximum likelihood estimates of the *Flexible Lomax* distribution. Table 3 and Table 5 represents the AIC, CAIC, BIC, and HQIC, which are goodness of fit criteria. In Tables 3 and 5 we see that the values of AIC, CAIC, BIC, and HQIC were the least for the *Flexible Lomax* distribution among the distributions tested, indicating the best fit with this one. Hence the FL was preferable with these data sets over *Lomax*, *Exponential Lomax, Weibull Lomax* and *POLO* distributions.

12. Total Time on a Test (TTT)

The TTT plot plays an important role in identifying the appropriate model to fit given data on failure rates. This plot tells us the different forms of the failure rate. If the TTT plot shows a straight line (diagonal), this indicates a constant failure rate. The failure rates will increase if this plot is concave, and decrease if it is convex. With a bath-tub shape, this plot first decreases and then increases. Similarly, if the failure rates follow an inverted bath-tub shape, then the curve is first concave and then convex. The TTT plot is determined by the following formula

Table 2. Maximum likelihood estimates.

Model	Estimates			
$FL(a,b,c)$	0.09592249	1.78995495	9.98465605	–
$POLO(a,b,c)$	0.1721965	14.4944336	4.9109785	–
$EL(a,b,c)$	28.842426	1.481920	2.482791	–
$WL(a,b,c,d)$	2.8345778	1.9742578	1.0284592	0.2073842
$L(a,b)$	2.259102	13.107217	–	–

Table 3. Goodness of fit Criteria: AIC, CAIC, BIC, HQIC.

Model	*AIC*	*CAIC*	*BIC*	*HQIC*
$FL(a,b,c)$	229.9754	230.6611	234.9661	231.766
$POLO(a,b,c)$	236.8627	237.5484	241.8534	238.6533
$EL(a,b,c)$	237.7877	238.4734	242.7784	239.5783
$WL(a,b,c,d)$	249.5339	250.7104	256.1881	251.9214
$L(a,b)$	252.6833	253.0166	256.0104	253.877

Table 4. Maximum likelihood estimates.

Model	Estimates			
$FL(a,b,c)$	8.517311	5.811554	3.009885	–
$POLO(a,b,c)$	1.629216	26.053456	1.458921	–
$EL(a,b,c)$	3.8661	28.4134	–	–
$WL(a,b,c,d)$	0.993276790	0.044379520	0.004883925	–
$L(a,b)$	5.4732949	1.5096438	4.7310404	0.2643016

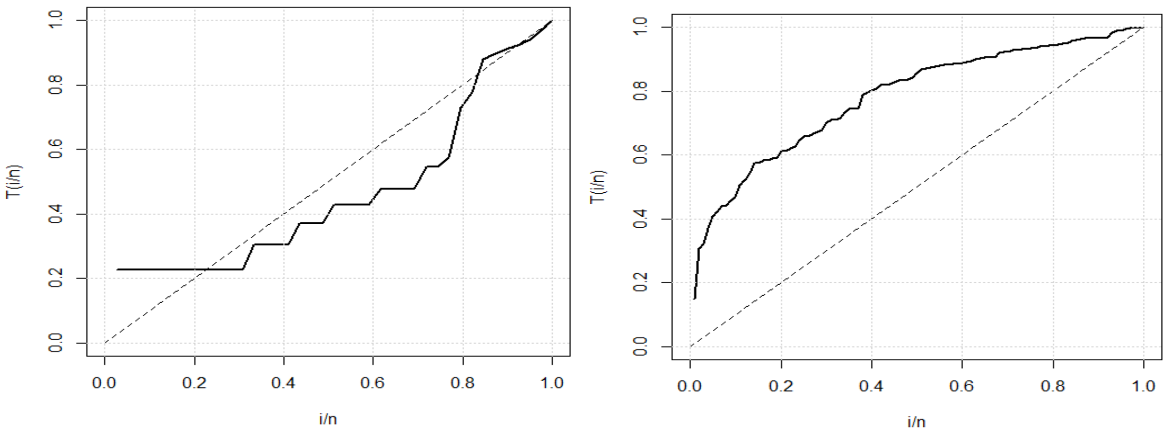
Table 5. Goodness of fit Criteria: AIC, CAIC, BIC, HQIC.

Model	*AIC*	*CAIC*	*BIC*	*HQIC*
$FL(a,b,c)$	288.5164	288.7664	296.3319	291.6794
$POLO(a,b,c)$	826.0647	826.2582	834.6208	829.5411
$EL(a,b,c)$	835.54	835.64	841.25	837.86
$WL(a,b,c,d)$	836.079	836.2725	844.635	839.5553
$L(a,b)$	828.6928	829.018	840.1009	833.328

$$G\left(\frac{r}{n}\right)=\frac{\sum_{i=1}^{r} x_{i:n}+(n-r)x_{i:n}}{\sum_{i=1}^{n} x_{i:n}}, \quad r=x_{i:n}=1,2,3,\ldots n \tag{12.1}$$

where $x_{i:n}$ are the order statistics against r/n.

The TTT plots for the data *Losses due to wind catastrophes* and for *breaking stresses of carbon fibers* are given in Figure 3. The graphs clearly show that the proposed distribution has importance for cases with both monotonic and non-monotonic hazard rates.



Losses due to wind catastrophes Breaking stress of carbon fibers

Figure 3. Hazard rate function of the Flexible Lomax.

13. Conclusions

In this paper, we presented a new modification with three parameters of the *Lomax* distribution, called the *Flexible Lomax* distribution (FL). Associated statistical functions of the FL distribution were obtained, like the hazard function, the survival function, mode, order statistics, etc. Furthermore, the model parameters can be estimated with the maximum likelihood approach. A simulation study was formulated and run, showing that bias and mean square error decrease with sample size for the FL distribution. The application of the FL distribution was then demonstrated using two real data sets, with AIC, CAIC, BIC, and HQIC criteria. In these lifetime data sets the FL distribution outperformed the other distributions tested. Hence, we conclude that the FL distribution is more flexible than *Lomax*, *Exponential Lomax*, and *POLO* distributions, and is expected to perform comparatively well with appropriate data sets.

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