

*Original Article*

# Greedy strategy for some normal $m \times n$ closing octagons games and winning strategy for normal $1 \times n$ and $2 \times n$ closing octagons games

Ratinan Boonklurb\* and Thitiphut Leelathanakit

*Department of Mathematics and Computer Science, Faculty of Science,  
Chulalongkorn University, Pathum Wan, Bangkok, 10330 Thailand*

Received: 18 August 2019; Revised: 30 September 2019; Accepted: 14 November 2019

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**Abstract**

$m \times n$  Closing Octagons (CO) game is a combinatorial game for two players. The game starts with an  $m \times n$  array of octagons such that every two adjacent octagons has one common side and 0 points. Players alternately turn by the following rules. (i) A player moves by coloring one side of an octagon. (ii) A player who colors the eighth side of  $k$  octagons earns  $k$  points and gets one more move. The game ends when every side of the octagons has been colored and the player with the most points wins. This game is formulated into a new CO game using graphs. In order to analyze it, more rules are added and the game with these additional rules is called a normal  $m \times n$  CO game. In this article, we give a greedy strategy for some normal  $m \times n$  CO games and a winning strategy for normal  $1 \times n$  and  $2 \times n$  CO games.

**Keywords:** closing octagons game, combinatorial game, 2-person game, strategy

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**1. Introduction**

Closing Octagons (CO) game is a combinatorial game that Leelathanakit and Boonklurb (2018) modified from Dots and Boxes and Dots and Hexagons games. A Combinatorial game (Ferguson, 2014) is a 2-person game with perfect information and no chance moves. It is determined by a set of states, including the initial state, which is the state at the beginning of the game. The players usually take turns alternately. Play shifts from one state to another until the terminal state is reached. The terminal state is a state in which no more moves are possible. After that, one player is declared the winner and the other the loser. However, two players can draw if neither has won.

Dots and Boxes game (Berlekamp, 2000) is a combinatorial game that can be found even on the internet. The game starts with an  $(m + 1) \times (n + 1)$  array of dots and 0 points for both players. Players alternately draw a vertical or horizontal line connecting two adjacent dots. The player who draws the fourth line of  $k$  square boxes of size  $1 \times 1$  earns  $k$  points and draws one more line. The game ends when every

pair of two adjacent dots has a line, and the player having the most points wins. The game has been analyzed by several researchers. For example, Lanhardt (2008) studied  $1 \times n$  Dots and Boxes game, and Buzzard and Ciere (2014) gave a highly efficient algorithm for playing Dots and Boxes game optimally. Recently, Ratiprasit, Simadhamnard and Teera vichayangoon derived Dots and Hexagons game from Dots and Boxes game. They changed square boxes to hexagonal boxes and used the same rules of playing as in Dots and Boxes game.

Leelathanakit and Boonklurb (2018) actually formulated their CO game into the new CO game using a graph. They inserted some additional rules to the formulated CO game and the formulated CO game with the additional rules is called a normal game. They studied some preliminary results concerning this game and gave a winning strategy for a normal  $3 \times 3$  CO game.

In this article, we consider their formulated CO game using a graph that is normal and would like to find winning strategies for several situations of this game. In Section 2, we formulate the greedy strategy, which can be a winning strategy for some  $m \times n$  CO games. In Section 3, we give a strategy for a player to win or draw when playing the formulated normal  $1 \times n$  and  $2 \times n$  CO games. In the last section, we provide an example showing that there is a case

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\*Corresponding author

Email address: [ratinan.b@chula.ac.th](mailto:ratinan.b@chula.ac.th)

for which the greedy strategy is not a winning strategy and it will be our future work to find out some other strategy for that situation. Note that a *strategy* is a plan of playing the game, which is constructed for moves or turns of a player. However, it must be usable and not contradict the main rules of the game. A strategy is called a *winning strategy* for a player if the player wins when the player plays according to the plan of the strategy, no matter how the opponent plays.

**2. CO game: Its Formulation and Greedy Strategy**

The CO game has been introduced by Leelathanakit and Boonklurb (2018), however, for ease of reference, let us introduce this game again in this section. Moreover, definitions for several common graph notations and terminologies can be found, for example for vertex, edge, loop, multiple edges, path, connected graph, component, and graph isomorphism, in Gross and Yellen (1999) and West (2001).

Let  $m, n$  be positive integers. The *CO game* or  $m \times n$  *CO game* is a game for two players starting with an  $m \times n$  array of octagons such that every two adjacent octagons have one common side and both players have 0 points. The two players have alternating turns by the following rules.

- (1) A player moves by coloring one side of an octagon.
- (2) A player who colors the eighth side of  $k$  octagons earns  $k$  points and takes one more move.

The game ends when every side of all octagons has been colored and the player having the most points wins. Figure 1 shows the beginning and the ending of  $2 \times 2$  CO game.

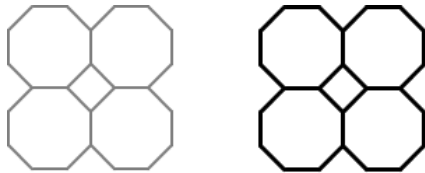


Figure 1. The beginning and the ending of  $2 \times 2$  CO game

Next, Leelathanakit and Boonklurb (2018) formulated the  $m \times n$  CO game into a new game using graphs. Let  $(G_0)_{m \times n}$  be a graph representing the  $m \times n$  array of octagons with no colored-sides by regarding the set of octagons to be the vertex set of  $(G_0)_{m \times n}$ , the edge set of  $(G_0)_{m \times n}$  is the set of sides of octagons such that  $e$  is an edge incident to vertices  $u$  and  $v$  called a *simple edge* if  $e$  is a common side of octagons  $u$  and  $v$  and  $e$  is a loop incident to a vertex  $v$  if  $e$  is an un-common side of an octagon  $v$ .

**Definition 2.1** (Leelathanakit & Boonklurb, 2018) Let  $m, n$  be positive integers. The *formulated  $m \times n$  CO game* is a game for two players starting with the graph  $(G_0)_{m \times n}$  and 0 initial points for both players. Two players alternately act by the following rules.

- (1') A player moves by removing one edge of the graph.
- (2') A player who removes the last edge of  $k$  vertices earns  $k$  points and takes one more move.

The game ends when the graph contains no edges and the player having the most points wins.

For example, Figure 2 shows the beginning of formulated  $2 \times 2$  CO game. It can be seen from this figure that the graph under consideration consists of simple edges and loops. There can be multiple loops at one vertex.

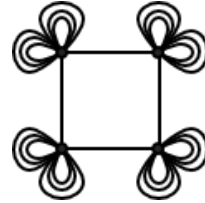


Figure 2. The beginning of formulated  $2 \times 2$  CO game

Note that, by Definition 2.1, a *move* is a removal of one edge of the graph, while a *turn* is a possible list of consecutive move(s) by one player that satisfies the rules of playing the game and  $m \times n$  is said to be the size of the game. It can be proved that the number of moves of an  $m \times n$  CO game is exactly  $6mn + m + n$ , see Lemma 2.1 of Leelathanakit and Boonklurb (2018). Henceforth, in this manuscript, the CO game refers to the formulated game using graphs.

Now, there are some notations and terminology for this game that we would like to introduce.

**Definition 2.2** (i) (Leelathanakit & Boonklurb, 2018) For any  $m \times n$  CO game, a *state*  $S_i$  is a triple  $(G(S_i), p_1, p_2)$  where  $(G(S_i), p_1,$  and  $p_2$  are a graph and points of Player I and Player II that are changed by turns, respectively, including  $(G(S_0), 0, 0)$  where  $G(S_0) = (G_0)_{m \times n}$ . Note that for positive integer  $i, S_i$  is the state obtained by a turn from  $S_{i-1}$ .  $S_0$  is called the *initial state* and a state that no moves are possible is called the *terminal state*.

(ii) (Leelathanakit & Boonklurb, 2018) A non-terminal state  $S$  of an  $m \times n$  CO game is a *normal state* if there is no possible turn giving points to a player from  $S$  to another state, otherwise  $S$  is called *strategic state*.

(iii) (Leelathanakit & Boonklurb, 2018) For a state  $S$ , a component  $K$  of  $G(S)$  is called a *weak component* if there is a possible turn from  $S$  to another state such that all edges of  $K$  are removed, otherwise  $K$  is called a *strong component*.

**Example 2.1** Consider the  $2 \times 2$  CO game that is shown in Figure 3.

- (i) It has 26 states, 25 turns and two players draw.
- (ii)  $S_1, S_2, S_3, \dots, S_{18}, S_{20}, S_{21}, S_{22}$  and  $S_{23}$  are normal states and  $S_{19}$  and  $S_{24}$  are strategic states.

Next, let us introduce some graph terms, namely,  $k$ -bouquet graph and  $k$ -pseudopath.

**Definition 2.3** (Gross & Yellen, 1999) (i) Let  $k$  be a positive integer. A *bouquet* or  $k$ -*bouquet graph* is a graph having exactly 1 vertex and  $k$  loops.

(ii) Let  $k$  be a positive integer. A connected graph  $G$  with  $k + 1$  vertices is a *pseudopath* or  $k$ -*pseudopath* if  $G$  contains a  $k$ -path subgraph  $P$  such that for each edge  $e$  of  $G, e$  is either a loop of  $G$  or an edge of  $P$  (Figure 4-5).

We can see that if some player turns from a normal state to a strategic state, then the opponent can earn some points from this strategic state. Thus, in general, if there is a possible turn from a normal state to another normal state, then

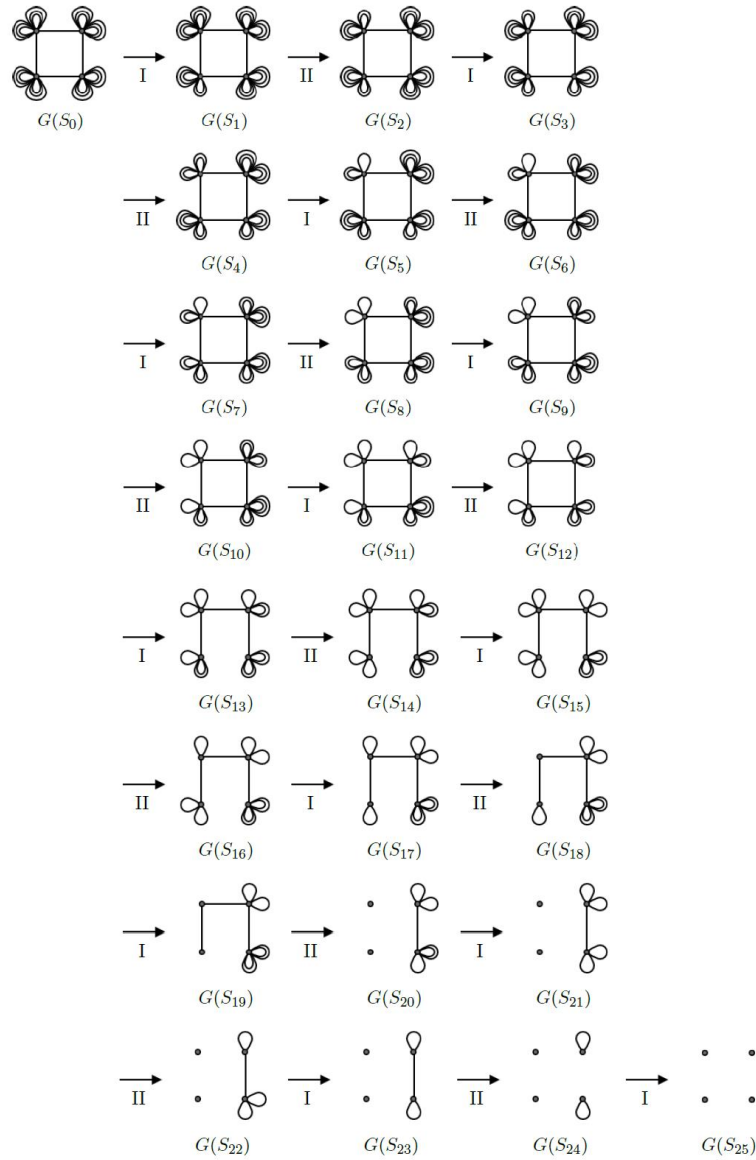


Figure 3. A 2 x 2 CO game

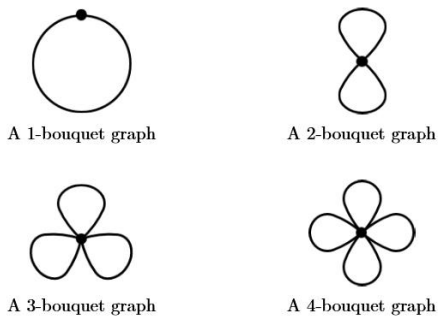


Figure 4. Bouquet graphs

players often make the turn. Moreover, if all components of the graph of a strategic state are weak components, then players often remove all edges of the graph. In order to

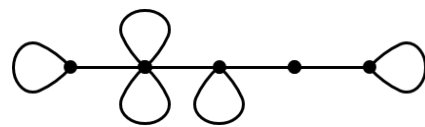


Figure 5. A pseudopath

analyze a winning strategy, we add more rules into this game and the game with these additional rules is called a normal  $m \times n$  CO game. The following definition is slightly different from the one that appeared in Leelathanakit and Boonklurb (2018).

**Definition 2.4** An  $m \times n$  CO game is a normal  $m \times n$  CO game if two players turn by the following rules.

(3') If there is a possible turn  $\tau$  from a state to a normal state, then a player has to make the turn  $\tau$ .

(4') If all components of the graph  $G(S)$  of a state  $S$  are weak components, then a player has to remove all edges of  $G(S)$ .

For a normal  $m \times n$  CO game, a normal state  $S$  of is a *critical state* if there is no possible turn from  $S$  to another normal state.

**Remark 2.1** (i) Leelathanakit and Boonklurb (2018) proved that every normal  $m \times n$  CO game has exactly one critical state.

(ii) Let  $S$  be a normal state which is not the terminal state of a normal  $m \times n$  CO game and  $G(S)$  have exactly  $k$  edges. Then, Theorem 2.3 of Leelathanakit and Boonklurb (2018) determines that if  $k - m - n$  is even, then a turn from  $S$  to another state is Player I's and if  $k - m - n$  is odd, then a turn from  $S$  to another state is Player II's.

Before we give specific strategies for normal  $1 \times n$  and  $2 \times n$  CO games, let us provide a basic greedy strategy for a normal  $m \times n$  CO game that seems to be a common sense.

**Strategy 2.1 (Greedy Strategy)** Let  $S$  be a state of an  $m \times n$  CO game. A player has to make a maximum turn from  $S$  to another state.

**Theorem 2.1** For playing normal  $m \times n$  CO game,

(i) if  $m$  and  $n$  are odd and all components of the graph of the critical state are 2-bouquets, then Strategy 2.1 is a winning strategy for Player II, and

(ii) if  $m$  or  $n$  is even and all components of the graph of the critical state are 2-bouquets, then a player who uses Strategy 2.1 wins or draws.

**Proof.** (i) Let  $S_k$  be the critical state and Player II play according to Strategy 2.1. Then,  $G(S_k)$  has exactly  $2mn$  edges. Obviously,  $2mn - m - n$  is even. By Remark 2.1 (ii), a turn from  $S_k$  to  $S_{k+1}$  is Player I's. Since all  $mn$  components of  $G(S_k)$  are 2-bouquets, Player I turns from  $S_k$  to  $S_{k+1}$  by removing a loop of a 2-bouquet component. Then,  $G(S_{k+1})$  has exactly one 1-bouquet component and  $mn - 1$  2-bouquet components. By Strategy 2.1, Player II turns from  $S_{k+1}$  to  $S_{k+2}$  by removing a loop of the 1-bouquet component and a loop of a 2-bouquet component, respectively. Then, Player II earns 1 point and  $G(S_{k+2})$  has exactly one 1-bouquet component and  $mn - 2$  2-bouquet components.

Next, it is easy to see that two players alternately turn from  $S_{k+2}$  to  $S_{k+mn}$  such that Player I either turns by removing a loop of a 2-bouquet component or turns by removing a loop of a 1-bouquet component and a loop of a 2-bouquet component, respectively, and earns at most 1 more point, and Player II turns by removing all edges of 1-bouquet components and a loop of a 2-bouquet component, respectively, and earns at least 1 more point.

Now, Player II turns from  $S_{k+mn}$  to the terminal state  $S_{k+mn+1}$  by removing all edges of 1-bouquet components. Then, Player II earns at least 1 more point. Since  $mn$  is odd and the first point is of Player II, Player II can earn at least  $\frac{mn+1}{2}$  points and Player I can earn at most  $\frac{mn+1}{2}$  points. Therefore, Player II wins.

(ii) Similar to the proof of (i), a player who uses Strategy 2.1 can earn at least  $\frac{mn}{2}$  points and the opponent can earn at most

$\frac{mn}{2}$  points.

### 3. Winning Strategy for Playing Normal $1 \times n$ and $2 \times n$ CO games

In this section, we consider only the normal  $1 \times n$  and  $2 \times n$  CO games and analyze the winning strategy for playing these games. First of all, let us consider simple case for the normal  $1 \times 2$  CO game.

**Theorem 3.1** For any normal  $1 \times 2$  CO game, Player II wins or draws.

**Proof.** Let  $S_k$  be the critical state of a normal  $1 \times 2$  CO game. Then, there are two cases of  $G(S_k)$  up to isomorphism shown in Figure 6.



Figure 6. Two cases of the graph of the critical state of a normal  $1 \times 2$  CO game

**Case 1**  $G(S_k)$  is a 1-pseudopath. By Remark 2.1 (ii), a turn from  $S_k$  to  $S_{k+1}$  is Player I's. Then, a Player I's turn from  $S_k$  to  $S_{k+1}$  is removing an edge of  $G(S_k)$ . It is clear that all components of  $G(S_{k+1})$  are weak components. Thus, a Player II's turn from  $S_{k+1}$  to  $S_{k+2}$  is by removing all edges of  $G(S_{k+1})$ . This implies that Player II earns 2 points.

**Case 2** All components of  $G(S_k)$  are 2-bouquets. By Remark 2.1 (ii), a turn from  $S_k$  to  $S_{k+1}$  is Player II's. Then, Player II's turn from  $S_k$  to  $S_{k+1}$  is by removing a loop of a 2-bouquet component. Thus,  $G(S_{k+1})$  has exactly one 1-bouquet component and one 2-bouquet component. Then, Player I's turn from  $S_{k+1}$  to  $S_{k+2}$  is either by removing a loop of a 2-bouquet component or by removing a loop of a 1-bouquet component and a loop of a 2-bouquet component, respectively. Then, Player I earns at most 1 point and all components of  $G(S_{k+2})$  are weak components. Now, Player II's turn from  $S_{k+2}$  to  $S_{k+3}$  is by removing all loops of  $G(S_{k+2})$ . This implies that Player II can earn at least 1 point.

Therefore, both cases imply that Player II wins or draws.

**Theorem 3.2** For playing a normal  $1 \times 2$  CO game, there is a strategy for Player I to draw.

**Proof.** We construct a strategy for Player I as follows.

(A) Player I has to turn from  $S_0$  to  $S_1$  by removing a simple edge of  $G(S_0)$ .

(B) Let  $S_k$  be the critical state. Player I has to turn from  $S_{k+1}$  to  $S_{k+2}$  by removing a loop of a 1-bouquet component and a loop of a 2-bouquet component, respectively.

Then, (A) implies that all components of  $G(S_k)$  are 2-bouquets. By Case 2 of Theorem 3.1, (B) implies that Player I earns 1 point. Therefore, Player I draws.

Now, we are ready to give the strategy for playing normal  $1 \times n$  and  $2 \times n$  CO games.

**Strategy 3.1** Let  $S_i$  be a state of a  $1 \times n$  or  $2 \times n$  CO games, a player turns according to the following lists.

(A) The player has to turn from  $S_0$  to  $S_1$  by removing a simple edge.

(B) If  $S_i$  be a normal state such that  $i \neq 0$ , then

(B1) if the opponent's turn from  $S_{i-1}$  to  $S_i$  is by removing a loop incident to a vertex  $v$  and  $G(S_i)$  contains a simple edge  $e$  incident to  $v$ , then the player has to remove  $e$ ;

(B2) if the opponent's turn from  $S_{i-1}$  to  $S_i$  is by removing a loop incident to a vertex  $v$  and  $G(S_i)$  contains no simple edges incident to  $v$  but  $G(S_i)$  contains a simple edge  $e$ , then the player has to remove  $e$ ;

(B3) if the opponent's turn from  $S_{i-1}$  to  $S_i$  is by removing a simple edge and  $G(S_i)$  contains a simple edge  $e$ , then the player has to remove  $e$ ;

(B4) if  $G(S_i)$  contains no simple edges but  $G(S_i)$  contains a loop  $l$  such that removing  $l$  is a turn from  $S_i$  to a normal state, then the player has to remove  $l$ .

(C) If  $S_i$  is a strategic state, then the player has to turn by using Strategy 2.1.

**Theorem 3.3** For playing normal  $1 \times n$  CO game,

(i) if  $n$  is odd, then Strategy 3.1 is a winning strategy for Player II, and

(ii) if  $n$  is even, then a player who uses Strategy 3.1 wins or draws.

**Proof.** (i) Let Player II play according to Strategy 3.1. Then,  $G(S_0)$  has 2 vertices incident to 7 loops and 1 simple edge and  $n - 2$  vertices incident to 6 loops and 2 simple edges as shown in Figure 7.

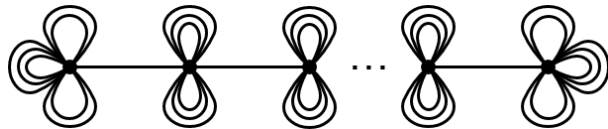


Figure 7. The graph of the initial state of a  $1 \times n$  CO game

For each vertex  $v$  of  $G(S_0)$ , the number of loops of  $G(S_0)$  incident to  $v$  is greater than the number of simple edges of  $G(S_0)$  incident to  $v$  by at least 4. By Strategy 3.1 (A), (B1), (B2) and (B3), each removing a simple edge is a turn from a normal state to another normal state and all simple edges have to be removed before the critical state is reached.

Then, we obtain that all components of the graph of the critical state are 2-bouquets. By Remark 2.1 (ii), a turn from the critical state to the first strategic state is Player I's. By Strategy 3.1 (C) and Theorem 2.1 (i), Player II wins.

(ii) Let us assume that a player plays according to Strategy 3.1. Similar to the proof of (i), all components of the graph of the critical state are 2-bouquets. We can see that Strategy 3.1 (B4) forces the player to make a maximum turn from this critical state. That is, the player plays according to Strategy 2.1. By Theorem 2.1 (ii), the player who uses Strategy 3.1 wins or draws.

**Theorem 3.4** For playing normal  $2 \times n$  CO game, a player who uses Strategy 3.1 wins or draws.

**Proof.** Let us assume that a player plays according to Strategy 3.1. Then,  $G(S_0)$  has 4 vertices incident to 6 loops and 2

simple edges and  $2n-4$  vertices incident to 5 loops and 3 simple edges as shown in Figure 8.

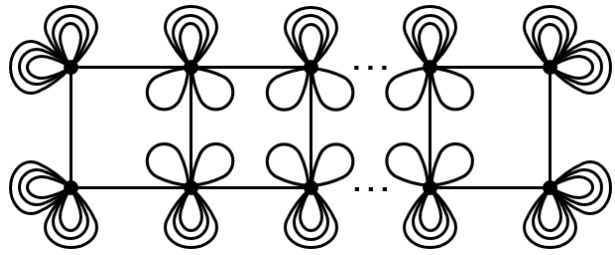


Figure 8 The graph of the initial state of a  $2 \times n$  CO game

For each vertex  $v$  of  $G(S_0)$ , the number of loops of  $G(S_0)$  incident to  $v$  is greater than the number of simple edges of  $G(S_0)$  incident to  $v$  by at least 2. By Strategy 3.1 (A), (B1), (B2) and (B3), each removing a simple edge is a turn from a normal state to another normal state and all simple edges have to be removed before the critical state is reached.

Then, we obtain that all components of the graph of the critical state are 2-bouquets. We can see that Strategy 3.1 (B4) forces the player to make a maximum turn from this critical state. That is, the player plays according to Strategy 2.1. By Theorem 2.1 (ii), the player who uses Strategy 3.1 wins or draws.

Notice that if there is a winning strategy for some player to play normal  $1 \times n$  CO game, where  $n$  is even, or normal  $2 \times n$  CO games and he plays according to it, then he wins no matter how the opponent plays. This contradicts Theorem 3.3 (ii) and Theorem 3.4 because the opponent can use Strategy 3.1 to win or draw. Therefore, we have the following corollary.

**Corollary 3.1** (i) For playing a normal  $1 \times n$  CO game, where  $n$  is even, then there is no winning strategy for both players.

(ii) For playing a normal  $2 \times n$  CO game, there is no winning strategy for both players.

#### 4. Conclusions

We consider a normal CO game using graph formulation and provide winning strategy for playing normal  $1 \times n$  and  $2 \times n$  CO games. In addition, we also give a basic greedy strategy for a normal  $m \times n$  CO game. We note that the greedy strategy may not be a good strategy in some situations. For example, consider a normal  $3 \times 3$  CO game such that the graph of the critical state  $S_k$  is shown in Figure 9.

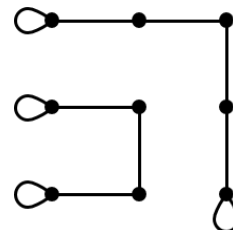


Figure 9. The graph of the critical state  $S_k$

Since  $G(S_k)$  has 11 edges, Remark 2.1 (ii) implies that a turn from  $S_k$  to  $S_{k+1}$  is Player II's. If Player II makes a turn from  $S_k$

to  $S_{k+1}$  that is shown in Figure 10, then  $G(S_{k+1})$  has exactly 2 weak components and 1 strong component.

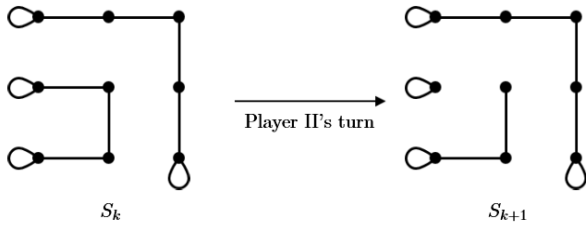


Figure 10. Player II's turn from  $S_k$  to  $S_{k+1}$

If Player I makes a turn from  $S_{k+1}$  to  $S_{k+2}$  by using greedy Strategy 2.1 that is shown in Figure 11, then Player I earns 4 points.

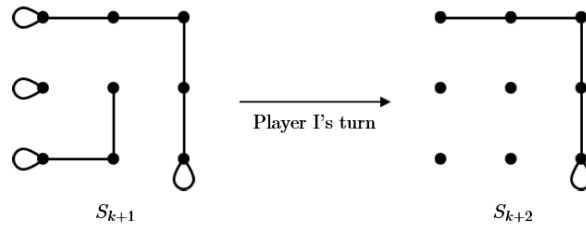


Figure 11. Player I's turn from  $S_{k+1}$  to  $S_{k+2}$

Next, Player II has to make a turn from  $S_{k+2}$  to the terminal state  $S_{k+3}$  by removing all edges of  $G(S_{k+2})$  and then Player II

earns 5 points. Therefore, Player I loses. As a result, our future work will be giving some other strategies that cover more situations and we may construct some winning strategies for other sizes of normal CO games.

**Acknowledgements**

We would like to thank the referees for carefully reading our manuscript and for giving comments which substantially helped improving the quality of the paper.

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