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Original Article

Bound conditions on *n*- polynomials whose coefficients, roots and critical points are integers

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Abstract

In this research we will provide the necessary conditions on the missing coefficients of polynomials of any degree so that roots and critical points are integers. Moreover, we completely determine all cubic polynomials whose coefficients, roots, and critical points are integers. Also the algorithm and source code to search all possible coefficients are provided.

Keywords: critical point, cubic polynomial, bound condition, n-polynomial, monic polynomial

1. Introduction

Finding the roots of higher degree polynomials is much more difficult than finding the roots of a quadratic polynomial. To make it easier, there are a few tools. Firstly, if r is a root of a polynomial equation, then $(x - r)$ is a factor of the polynomial (Burton, 1970; Rosen, 2011). Secondly, any polynomials with real coefficients can be written as the product of linear factors (of the form $(x - r)$) and quadratic factors which are irreducible over the real numbers. Finally, a quadratic factor that is irreducible over the real is a quadratic function with no real solutions; that is, its discriminant is negative. All factors, linear and quadratic, will have real coefficients. For more details, see (Barbeau, 2003; Rosen, 2011). Two other theorems also have to do with the roots of a polynomial, Descartes' Rule of Signs, and the Rational Root Theorem. Descartes' Rule of Signs has to do with the number of real roots possible for a given polynomial $f(x)$, (Barbeau, 2003). The Rational Root Theorem is another useful tool in finding the roots of a polynomial $y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$. If the coefficients of a polynomial are all integers, and a root of the polynomial is rational (it can be expressed as a fraction in lowest terms), the numerator of the root is a factor of a_0 and the denominator of the root is a factor of a_n. For more details, see (Barbeau, 2003; Rosen, 2011; Milovanovic, Mitrinovic & Rassias, 1994; Rahman & Schmeisser, 2002). In (Lossers, 1989), Lossers explained how to find integer roots of a cubic polynomial. Up to now, most of the tools here give the way to find the integer roots of polynomials. However, to find polynomials whose coefficients, roots, and critical points are integers is a more interesting problem. This problem is on the list of unsolved problems published by Richard Nowakowski (Nowakowski, 1999). Such polynomials are called nice polynomials. Thus, various techniques have been proposed to solve and attempt to complete the problem in many papers, (Bruggeman & Gush, 1980; Buddenhagen, 1992; Caldwell, 1990; Chapple, 1990; Carroll, 1989; Galvin, 1990; Groves, 2007a, 2007b, 2008c). But in all cases all coefficients must be known. However, if some coefficients are known and some are missing, how can we find all missing coefficients so that polynomials are nice polynomials? This is also an interesting problem. For the quadratic polynomial, it is not difficult to determine missing coefficients since the quadratic formula will come to play. However, there is no implicit method for finding missing coefficients of the polynomials in the higher degree. Thus, in this research we will provide the necessary conditions on the missing

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coefficients of polynomial of any degree so that roots and critical points are integers. Moreover, we completely determine all cubic polynomials whose coefficients, roots, and critical points are integers. Finally, we will provide the algorithm and source code to search all possible coefficients.

Consider $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where $a_0, a_1, \ldots, a_n$ are integers. To find conditions on coefficients *p(x)* and *p'(x)* for which their roots are integers in general, we first state the very well-known Theorems:

Theorem 1.1.
Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree $n \geq 1$ where $a_0, a_1, \ldots, a_n$ are integers. Then
1) The polynomial $p(x)$ has exactly n roots, counting multiplicities, and
2) $p(x) = a_n(x - r_1)(x - r_2)\cdots(x - r_n)$, where $r_1, r_2, \ldots, r_n$ are the roots of $p(x)$.

Theorem 1.2.
Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree $n \geq 1$ where $a_0, a_1, \ldots, a_n$ are integers. Suppose that the rational number $r = \dfrac{k}{r_0}$, where $\gcd(k, r_0) = 1$, is a root of $p(x)$. Then, the integer k is a divisor of a_0 and the integer r_0 is a divisor of a_n. Moreover, if $a_n = 1$, then all rational roots are integers. Also, we need the following lemmas from Number Theory.

Lemma 1.3.
Suppose that a, b, c are positive integers, and that a is a divisor of the product bc. If $\gcd(a,b) = 1$, then a is a divisor of c.

If $r_1, r_2, \ldots, r_n$ are integer roots of $p(x)$, then by Theorem 1.1 we have

$$\begin{aligned} p(x) &= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \\ &= a_n(x - r_1)(x - r_2)\cdots(x - r_n). \end{aligned}$$

And so, it deduces the following fact:

$$\frac{(-1)^i a_{n-i}}{a_n} = \sum_{1 \leq j_1 < j_2 < \cdots < j_i \leq n} (r_{j_1} r_{j_2} \cdots r_{j_i})$$

for all $j_i \in \{1, 2, \ldots, n\}$ and $i \in \{1, 2, \ldots, n\}$. (1)

Since all r_i are integers, it follows that a_n is a divisor of a_{n-i} for all $i \in \{1, 2, \ldots, n\}$. This means that the roots of $p(x)$ and $\dfrac{p(x)}{a_n}$ are the same. Moreover, by translation, we can investigate the polynomial all of whose roots and critical points are non-negative integers. Thus throughout this paper, it suffices to consider only the monic polynomial whose all roots and all critical points are non-negative integers.

Lemma 1.4.
Let $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where $a_0, a_1, \ldots, a_{n-1}$ are integers. If r and s are integer roots of $p(x)$ and *p'(x)*, respectively, then $r \mid a_0$ and $s \mid \dfrac{a_1}{n}$.

Proof. It follows from the equation (1).

Theorem 1.5.
Let $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_0, a_1, \ldots, a_{n-1}$ are integers. If all roots of $p(x)$ and *p'(x)* are integers, then n is a divisor of $a_{k_1}, a_{k_2}, \ldots, a_{k_{f(n)}}$, where $k_i \in \{k \mid 1 \leq k < n, \gcd(k, n) = 1\}$ and f is the Euler's totient function. If n is prime, then n is a divisor of $\gcd(a_1, a_2, \ldots, a_{n-1})$.

Proof. Since all roots of *p'(x)* are integers, it implies by the equation (1) that

$$\frac{(-1)^i (n-i) a_{n-i}}{n} = \sum_{1 \leq j_1 < j_2 < \cdots < j_i \leq n-1} (r_{j_1} r_{j_2} \cdots r_{j_i})$$

where r_{j_i}'s are roots of *p'(x)*.

Let $k_i \in \{k \mid 1 \leq k < n, \gcd(k, n) = 1\}$ for $i = 1, 2, \ldots, f(n)$. It follows that $\gcd(n, k_i) = 1$ and so n must divide a_{k_i} for $i = 1, 2, \ldots, f(n)$. In particular, if n is prime, then $\phi(n) = n - 1$ and hence n is a divisor of $\gcd(a_1, a_2, \ldots, a_{n-1})$.

Theorem 1.6.
Let $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_0, a_1, \ldots, a_{n-1}$ are integers and $a_0, a_1 \neq 0$. If all roots of $p(x)$ and *p'(x)* are non-negative integers, then

$$A_i \le (-1)^i a_{n-i} \le B_i \text{ for all } i \in \{1,2,\dots,n-2\} \qquad (2)$$

where

$$A_i = \max\left\{\binom{n}{i}\left(|a_0|\right)^{\frac{i}{n}}, \frac{n}{n-i}\binom{n-1}{i}\left(\frac{|a_1|}{n}\right)^{\frac{i}{n-1}}\right\}$$

and

$$B_i = \min\left\{\binom{n}{i}|a_0|, \frac{n}{n-i}\binom{n-1}{i}\left(\frac{|a_1|}{n}\right)\right\}.$$

Proof. Applying the equation (1) to $p(x) = 0$ and $p'(x) = 0$, we have

$$(-1)^i a_{n-i} = \sum_{1\le j_1<j_2<\cdots<j_i\le n} (r_{j_1} r_{j_2} \cdots r_{j_i})$$

for all $i \in \{1,2,\dots,n\}$

and

$$\frac{(-1)^i (n-i) a_{n-i}}{n} = \sum_{1\le j_1<j_2<\cdots<j_i\le n-1} (s_{j_1} s_{j_2} \cdots s_{j_i})$$

for all $i \in \{1,2,\dots,n-1\}$.

By AM-GM inequality, it implies that for all $i \in \{1,2,\dots,n\}$,

$$\sum_{1\le j_1<j_2<\cdots<j_i\le n} (r_{j_1} r_{j_2} \cdots r_{j_i}) \ge \binom{n}{i}\left(r_1 r_2 \cdots r_n\right)^{\frac{i}{n}} = \binom{n}{i}\left(|a_0|\right)^{\frac{i}{n}}$$

and for all $i \in \{1,2,\dots,n-1\}$,

$$\sum_{1\le j_1<j_2<\cdots<j_i\le n-1} (s_{j_1} s_{j_2} \cdots s_{j_i}) \ge \binom{n-1}{i}\left(s_1 s_2 \cdots s_{n-1}\right)^{\frac{i}{n-1}} = \binom{n-1}{i}\left(\frac{|a_1|}{n}\right)^{\frac{i}{n-1}}.$$

Hence,

$$(-1)^i a_{n-i} \ge A_i$$

where $A_i = \max\left\{\binom{n}{i}\left(|a_0|\right)^{\frac{i}{n}}, \frac{n}{n-i}\binom{n-1}{i}\left(\frac{|a_1|}{n}\right)^{\frac{i}{n-1}}\right\}$.

On the other hand, since r_i and s_j are all positive integers, it follows that

$$\sum_{1\le j_1<j_2<\cdots<j_i\le n} (r_{j_1} r_{j_2} \cdots r_{j_i}) \le \sum_{j=1}^{\binom{n}{i}} (r_1 r_2 \cdots r_n) = \binom{n}{i}|a_0|$$

and

$$\sum_{1\le j_1<j_2<\cdots<j_i\le n-1} (s_{j_1} s_{j_2} \cdots s_{j_i}) \le \sum_{j=1}^{\binom{n-1}{i}} (s_1 s_2 \cdots s_{n-1}) = \binom{n-1}{i}\frac{|a_1|}{n}.$$

Thus

$$(-1)^i a_{n-i} \le B_i$$

Where

$$B_i = \min\left\{\binom{n}{i}|a_0|, \frac{n}{n-i}\binom{n-1}{i}\frac{|a_1|}{n}\right\}.$$

This implies that $A_i \le (-1)^i a_{n-i} \le B_i$ for all $i \in \{1,2,\dots,n-2\}$.

Theorem 1.7.
Let $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_0, a_1, \dots, a_{n-1}$ are integers and $a_0, a_1 \ne 0$. If all roots of $p(x)$ and $p'(x)$ are non-negative integers, then

$$(-1)^i a_{n-i} \le \min\left\{(-1)^{n-i} a_i, \binom{i+1}{n-i}(-1)^{n-i-1} a_{i+1}\right\}$$

$$\text{for all } i \in \left\{1,2,\dots,\left\lfloor\frac{n}{2}\right\rfloor - 1\right\}.$$

(3)

Proof. Let $r_1, r_2, \dots, r_n$ and $s_1, s_2, \dots, s_{n-1}$ be non-negative integer roots of $p(x)$ and $p'(x)$, respectively. For each,

$i \in \left\{1,2,\dots,\left\lfloor\frac{n}{2}\right\rfloor - 1\right\}$, we have $\binom{n}{i} = \binom{n}{n-i}$

and $\binom{n-1}{i} = \binom{n-1}{n-1-i}$.

It follows that

$$\sum_{1\le j_1<j_2<\cdots<j_i\le n} (r_{j_1} r_{j_2} \cdots r_{j_i}) \le$$

$$\sum_{1\le j_1<j_2<\cdots<j_{n-i}\le n}(r_{j_1}r_{j_2}\cdots r_{j_{n-i}})$$

This implies that

$$(-1)^i a_{n-i} \le (-1)^{n-i} a_i \text{ and}$$

$$(-1)^i a_{n-i} \le \left|\frac{i+1}{n-i}\right| (-1)^{n-i-1} a_{i+1}.$$

Therefore

$$(-1)^i a_{n-i} \le \min\left\{(-1)^{n-i} a_i, \left|\frac{i+1}{n-i}\right| (-1)^{n-i-1} a_{i+1}\right\}.$$

With the approximation described above, it turns out that we can approximate the bound condition not only of the missing coefficients but also of the known coefficients of the polynomial. However, finding all missing coefficients of polynomials that have all integer roots and integer critical points is not easy and, without a computer, it also seems impossible. So, the next step is to develop the algorithm, by using conditions above in order to generate all missing coefficients in the polynomials such that all roots and critical points are integers.

2. The Cubic Polynomials

Now we completely find all the cubic polynomials whose coefficients, roots and critical points are integers. Also, we do not restrict to positive roots and positive critical points.

We first start with the cubic polynomial $p(x) = x^3 + bx^2 + cx + d$, where b,c,d are integers, and its derivative is $p'(x) = 3x^2 + 2bx + c$. Since $3 \mid \gcd(b,c)$, the polynomial $p'(x)$ can be reduced to $\overline{p'(x)} = \frac{p'(x)}{3} = x^2 + 2k_1 x + k_2$ for some $k_1, k_2 \in \mathbb{Z}$ and $b = 3k_1$ and $c = 3k_2$. Let p,q,r be solutions of $p(x) = 0$. Then

$$p(x) = x^3 + bx^2 + cx + d = (x - p)(x - q)(x - r).$$

Equating the coefficients, we have

$$p + q + r = -b,$$

$$pq + qr + rp = c,$$

$$pqr = -d.$$

It follows that $p + q = -b - r$ and $pq = c + br + r^2$.

Let m, m' be the roots of $\overline{p'(x)} = 0$. Then

$$m = \frac{-2k_1 + \sqrt{4k_1^2 - 4k_2}}{2} = -k_1 + \sqrt{k_1^2 - k_2}$$

and

$$m' = \frac{-2k_1 - \sqrt{4k_1^2 - 4k_2}}{2} = -k_1 - \sqrt{k_1^2 - k_2}.$$

Since $m = -k_1 + \sqrt{k_1^2 - k_2}$, we have $k_2 = -2mk_1 - m^2$ and hence

$$c = 3k_2 = -6mk_1 - 3m^2 = -2mb - 3m^2.$$

Therefore,

$$pq = c + br + r^2 = r^2 + br - 2mb - 3m^2.$$

At this point we can use the fact that any one of five roots can be equal to 0 by the translation. Without loss of generality let $m = 0$. Thus $pq = r^2 + br$. Note that the integers, p,q are also solutions to

$$x^2 - (-b - r)x + (r^2 + br) = x^2 - (p + q)x + pq = 0$$

and so $(p + q)^2 - 4pq = w^2$ for some integer w. Thus, $w^2 = b^2 + 2br + r^2 - 4r^2 - 4br = b^2 - 2br - 3r^2 = (b - 3r)(b + r)$.

Let $e = b + r = 3k_1 + r$ and then we can simplify that

$$w^2 = 3e(4k_1 - e). \tag{4}$$

To find k_1, e, w that satisfy the equation (4), we need to define the function $v(x)$, which is the smallest positive integer whose square is divisible by x. If $x = p_1^{a_1} p_2^{a_2} \ldots p_n^{a_n}$, then it is straight forward to check that

$$v(x) = p_1^{\left\lceil\frac{\alpha_1}{2}\right\rceil} p_2^{\left\lceil\frac{\alpha_2}{2}\right\rceil} \cdots p_n^{\left\lceil\frac{\alpha_n}{2}\right\rceil}.$$

To solve the equation (4), we consider 2 cases.

First case: $4 \mid e$. Then $e = 4k$ for some $k \in \mathbb{N}$. Thus, $w^2 = 3e(4k_1 - 4k) = 12e(k_1 - k)$. Note that $v(12e)$ $v(12e) = 2v(3e)$ and $2v(3e) \mid w$. So, $w = 2tv(3e)$ for some integer t. It follows that $k_1 = \frac{\frac{w^2}{3e} + e}{4}$ yields an integral solution.

Second case: $4 \nmid e$. Note that $v(3e) \mid w$. Then if $w = v(3e)$, then $\frac{w^2}{3e}$ is an integer and so letting $k_1 = \frac{\frac{w^2}{3e} + e}{4}$. If $w = tv(3e)$ for some integer t, it is not difficult to see that $k_1 = \frac{\frac{w^2}{3e} + e}{4}$ is an integer if and only if t is odd. Combining the cases, we find that the equation (4) has an integer solution if and only if

$$w = sv(3e) \tag{5}$$

where s is even if $4 \mid e$ and s is odd if $4 \nmid e$. Therefore, given $e \in \mathbb{N}$, $t \in \mathbb{Z}$, if $4 \mid e$, we have

$$b = 3\left(\frac{\frac{(2tv(3e))^2}{3e} + e}{4}\right) = \frac{\frac{(2tv(3e))^2}{e} + 3e}{4},$$

$$c = 0,$$

$$d = -e(e-b)^2 = -\frac{(e^2 - (2tv(3e))^2)^2}{16e}$$

and if $4 \nmid e$, we have

$$b = 3\left(\frac{\frac{((2t+1)v(3e))^2}{3e} + e}{4}\right) = \frac{\frac{((2t+1)v(3e))^2}{e} + 3e}{4},$$

$$c = 0,$$

$$d = -e(e-b)^2 = -\frac{(e^2 - ((2t+1)v(3e))^2)^2}{16e}.$$

It remains to check that p, q, r, m, m' are integers. Since $m = 0$ and $3 \mid b$, it follows that m' is an integer. By the quadratic formula, we have p and q are of the form $\frac{b+r+w}{2} = \frac{e+w}{2}$ or the form $\frac{b+r-w}{2} = \frac{e-w}{2}$. By the equation (5), we can see that w and e have the same parity check which implies that $\frac{e \pm w}{2}$ are always integers. That is, p and q are integers and since $p + q = -b - r$, it follows that r is an integer. Therefore, all roots and critical points of $p(x) = x^3 + bx^2 + cx + d$ are integers.

Denote l horizontal displacement and now we complete the proof of the following theorem.

Theorem 2.1. A monic cubic polynomial whose coefficients, roots, and critical points are integers is of the form

$$(x+l)^3 + \left(\frac{\frac{(sv(3e))^2}{e} + 3e}{4}\right)(x+l)^2 - \frac{(e^2 - (sv(3e))^2)^2}{16e}$$

where $l, s \in \mathbb{Z}$, $e \in \mathbb{N}$ *and* s *is even if* $4 \mid e$ *and* s *is odd if* $4 \nmid e$.

Corollary 2.2 Let $a_0 \in \mathbb{Z}$. Then a monic polynomial $p(x) = x^3 + a_2x^2 + a_1x + a_0$, where $a_1, a_2 \in \mathbb{Z}$ has roots and critical points are integers if and only if there are $l, e, s \in \mathbb{Z}$ such that

$$a_0 = l^3 + l^2\left(\frac{\frac{(sv(3e))^2}{e} + 3e}{4}\right) - \frac{(e^2 - (sv(3e))^2)^2}{16e},$$

$$a_1 = 3l^2 + 2l\left(\frac{\frac{(sv(3e))^2}{e} + 3e}{4}\right),$$

and

$$a_2 = 3l + \left(\frac{\frac{(sv(3e))^2}{e} + 3e}{4}\right).$$

Proof. This follows directly from Theorem 2.1.

For example, let $e = 8, s = 2, l = 0,$ we have $p(x) = x^3 + 24x^2 - 2048$ and its roots are $8, -16, -16.$ Also, its derivative is $3x^2 + 48x$ and its roots are 0 and -16.

3. Some Numerical Results of Polynomials with Higher Degree

In order to complete this study, an algorithm is given in the form of Octave command for finding all missing integer coefficients of polynomial whose roots and critical points are integers when some coefficients are known. For given positive integers $n, a_0,$ and $a_1,$ let us consider a monic polynomial

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

where $a_1, a_2, \ldots, a_n$ are integers. By Theorem 1.6 and Theorem 1.7, we know that

$$A_i \le (-1)^i a_{n-i} \le B_i \text{ for all } i \in \{1, 2, \ldots, n-2\}$$

where

$$A_i = \max\left\{\binom{n}{i}\left(|a_0|\right)^{\frac{i}{n}}, \frac{n}{n-i}\binom{n-1}{i}\left(\frac{|a_1|}{n}\right)^{\frac{i}{n-1}}\right\},$$

$$B_i = \min\left\{\binom{n}{i}|a_0|, \frac{n}{n-i}\binom{n-1}{i}\frac{|a_1|}{n}\right\},$$

and

$$(-1)^i a_{n-i} \le \min\left\{(-1)^{n-i}a_i, \left(\frac{i+1}{n-i}\right)(-1)^{n-i-1}a_{i+1}\right\} \text{ for all } i \in \left\{1, 2, \ldots, \left\lfloor\frac{n}{2}\right\rfloor - 1\right\}.$$

So the function "FindCoefficient (n, a_0, a_1)" is used to locate all a_i that satisfy the above conditions and screen those a_i out so that the function has integer roots and integer critical points.

```
function sol = FindCoefficient (n, a0, a1)
a1 = a0;
a2 = a1;
a(n+1) = 1;
for i = 1 : n - 2
A = max {nchoosek(n,i)*(abs(a0))^(i/n), (n/(n - i))*nchoosek(n - 1,i)*(abs(a1)/n)^(i/(n-1))};
B = min(nchoosek(n,i)*(abs(a0)), (n/(n - i))*nchoosek(n - 1,i)*(abs(a1)/n));
    if (mod(i,2) == 1)
C = A;
A = - B;
B = - C;
    end
if (length(A : B) > 0)
    a(n-i+1) = A : B;
else
a(n-i+1) = 0;
end
end
vout = cell(size(a));
[vout:] = ndgrid(a :);
for i = 1 : size(vout,2)
temp = vout(i);
p(:,i) = temp(:);
end
sol = [];
p = fliplr(p);
e = 0.001;
for i = 1 : size(p,1)
k = n : - 1 : 1;
q = p(i,1 : end - 1).*k;
s = roots(p(i,:));
ss = roots(q);
    if (sum(abs(s – round(s))==0) == length(s) &
sum(abs(ss – round(ss))==0) == length(ss) &
sum(imag(s)==e) == length(s) &
sum(imag(ss)==e) == length(ss))
sol = [sol; p(i,:) ];
end
end
```

For example, for $n = 3, -500 \leq a_0 \leq -200,$ and $0 \leq a_1 \leq 200$ all possible a_i are shown in Table 1 below:

Table 1. All possible a_i for $n = 3$ so that $p(x) = 0$ has integer roots and integer critical points.

a_0	- 490	- 486	- 432	- 425	- 400	- 352	- 350	- 343	- 324	- 320
a_1	189	189	180	195	180	192	165	147	144	144
a_2	- 24	- 24	- 24	- 27	- 24	- 30	- 24	- 21	- 21	- 21

4. Conclusions

Normally, to roughly sketch the graph of a polynomial $p(x)$ with degree n by hand, at least we need to know the x-intercepts and relative maxima or relative minima. That is, it requires solving the roots and critical points of $p(x) = 0$ and $p'(x) = 0$, respectively. However, for the higher degree, the equations $p(x) = 0$ and $p'(x) = 0$ is hard to solve unless their roots and critical points are integers. It is also not easy to illustrate examples of polynomials whose coefficients, roots, and critical points are integers without a computer auxiliary program. Moreover, it raises more problems in terms of time-consuming and existence if some coefficients of the polynomial are fixed. So, our work completely determines all cubic polynomials whose coefficients, roots, and critical points are integers and gives bound conditions in order to construct desired polynomials. Furthermore, we can take advantage from this work to reduce the amount of work and to increase the speed of a computer search for any higher degree of desired polynomials.

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