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Original Article

Numerical solution of time-fractional Benjamin-Bona-Mahony-Burgers equation via finite integration method by using Chebyshev expansion

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Abstract

The finite integration method using Chebyshev polynomial (FIM-CBS) has been proposed in order to overcome the difficulty of solving linear partial differential equations. In this paper, we develop the FIM-CBS in order to devise a powerful numerical algorithm for finding approximate solutions of the nonlinear time-fractional Benjamin-Bona-Mahony-Burgers equations with the initial and boundary conditions. The time-fractional derivative is in the Caputo sense which is estimated by the forward difference quotient. Furthermore, we implement our proposed algorithm via several numerical experiments by comparing the approximate results obtained by our method and other methods with their analytical solutions. It can be evidence that the developed FIM-CBS algorithm is very effective and efficient with a small number of computational grid points which is discretized by the zeros of Chebyshev polynomial of a certain degree.

Keywords: finite integration method, Chebyshev expansion, time fractional derivative, Benjamin-Bona-Mahony-Burgers equation, Caputo fractional derivative

1. Introduction

The fractional differential equations (FDEs) are used in many fields of sciences and engineering. In 1695, Leibniz and L'Hopital firstly introduced the basic concept of FDEs in which the order of derivative can take an integer or rational number in the interval $[0,1]$ (Oldham & Spanier, 1974). The applications of FDE have been occurring in various real world physical problems, such as diffusion processes (Metzler & Klafter, 2000), oscillating dynamical systems (Agila, Baleanu, Eid, & Irfanoglu, 2016), thermal conductivity (Kumar, Singh, & Baleanu, 2017), rheological models (Yang, Gao, & Srivastava, 2017), quantum models (Laskin, 2011) etc. In order to better understand these problems as well as further apply them in practical life, it is important to find their solutions. However, it is very difficult to find them in the closed form of analytical solution. Therefore, the numerical methods play an essential role to solve these problems. Some efficient methods have emerged, such as Adomian decomposition method (Li & Wang, 2009), homotopy perturbation method (Khan, Ara, & Mahmood, 2015), and variational iteration method (Yulita, Nooran, & Hashim, 2009).

Recently, Boonklurb, Duangpan, and Treeyaprasert (2018) have proposed the finite integration method using Chebyshev expansion (FIM-CBS), which uses Chebyshev polynomials to modify the original FIM (Wen, Hon, Li, & Korakianitis, 2013) for solving linear partial differential equations (PDEs). The FIM-CBS also provides a much higher accuracy than the finite difference method and those traditional FIMs with less computation. During recent year, many articles have successfully demonstrated the accuracy of FIMs for seeking numerical solutions both of linear and nonlinear PDEs. However, the application of FIM has not been performed for solving the nonlinear FDEs like the Benjamin-Bona-Mahony-Burgers (BBMB) equations. Originally, the BBMB equation was introduced by Benjamin, Bona, and Mahony (1972). It is the mathematical model of propagation for small amplitude long waves in nonlinear dispersive media system which was improved from the Korteweg-de Vries (KdV) equation. Normally, the BBMB and KdV equations are relevant to the wave breaking models, since the KdV model came from water

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waves which was used for long waves in many other physical systems. However, in some physical systems of long waves, the KdV equation was not applicable. Hence, the BBMB was proposed. It described the unidirectional transmission of long wave signals in a certain nonlinear dispersive system (Kondo & Webler, 2016). For the time-fractional BBMB equation, it was presented to discuss the dynamic behavior of physical systems (Kumar & Kumar, 2014).

Therefore, the time-fractional BBMB equations are operated in this paper by using the FIM-CBS proposed by Boonklurb *et al.* (2018) to construct a numerical algorithm for finding their approximate solutions. For the nonlinear term, we use the idea of Chebyshev expansion to handle it. The time fractional derivative term is in the Caputo sense, which estimated by the forward difference quotient. The rest of this paper is organized into four parts as follows. In Section 2, we develop the FIM-CBS to construct the Chebyshev integration matrix for the general range. Section 3 presents the numerical scheme by using the developed FIM-CBS for finding approximate solutions of the time-fractional BBMB equations. In Section 4, our numerical algorithm is evaluated for its effectiveness and accuracy via several experiments by comparing our numerical results with their exact solutions. We also illustrate the error and graph of their approximate solutions in the Section 4. Finally, conclusions and some discussions about this work and the future work are provided in Section 5.

2. Developed FIM-CBS in the General Range

In this section, we give some definitions and essential properties of Chebyshev polynomial: Mason and Handcomb (2013) for more details and proofs. Moreover, we also use the Chebyshev polynomials in the general range to develop the Chebyshev integration matrix of FIM-CBS. Then, our developed FIM-CBS can be applied to invent the numerical algorithm for performing on an arbitrary domain $[a, b]$ rather than $[-1,1]$.

Definition 2.1. The Chebyshev polynomial of the first kind of degree $n \geq 0$ is defined by

$$T_n(x) = \cos(n \cos^{-1} x) \quad \text{for} \quad x \in [-1,1].$$

However, The Chebyshev polynomial in the general range of degree n can be defined by

$$R_n(x) = T_n\left(\frac{2x-a-b}{b-a}\right) \quad \text{for} \quad x \in [a, b].$$

Henceforward, in this article, the Chebyshev polynomial of degree n refers to $R_n(x)$.

Lemma 2.2. (i) For $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$, the zeros of Chebyshev polynomial $R_n(x)$ are

$$x_k = \frac{1}{2}\left((b-a)\cos\left(\frac{2k-1}{2n}\pi\right) + a + b\right), \quad k \in \{1, 2, 3, \dots, n\}.$$

(ii) For $x \in [a, b]$, the single layer integrations of Chebyshev polynomial $R_n(x)$ are

$$\begin{aligned} \bar{R}_0(x) &= \int_a^x R_0(\xi)\, d\xi = x - a, \\ \bar{R}_1(x) &= \int_a^x R_1(\xi)\, d\xi = \frac{(x-a)(x-b)}{b-a}, \\ \bar{R}_n(x) &= \int_a^x R_n(\xi)\, d\xi = \frac{b-a}{4}\left(\frac{R_{n+1}(x)}{n+1} - \frac{R_{n-1}(x)}{n-1} - \frac{2(-1)^n}{n^2-1}\right), \quad n \geq 2. \end{aligned} \tag{1}$$

(iii) Let $\{x_1, x_2, x_3, \dots, x_n\}$ be the zeros of $R_n(x)$ and define the Chebyshev matrix $\mathbf{R}$ by

$$\mathbf{R} = \begin{bmatrix} R_0(x_1) & R_1(x_1) & \cdots & R_{n-1}(x_1) \\ R_0(x_2) & R_1(x_2) & \cdots & R_{n-1}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ R_0(x_n) & R_1(x_n) & \cdots & R_{n-1}(x_n) \end{bmatrix}. \tag{2}$$

Then, it has the multiplicative inverse $\mathbf{R}^{-1} = \frac{1}{n}\operatorname{diag}(1, 2, 2, \dots, 2)\mathbf{R}^T$.

To construct the Chebyshev integration matrix. We first let $M \in \mathbb{N}$ and $a, b \in \mathbb{R}$. Define an approximate solution $u(x)$ of a certain PDE by using a linear combination of Chebyshev polynomial $R_n(x)$, i.e.,

$$u(x) = \sum_{n=0}^{M-1} c_n R_n(x) \quad \text{for} \quad x \in [a, b],$$

where c_n is an unknown constant. For $k \in \{1, 2, 3, \dots, M\}$, let x_k be computational nodal points which are generated by the zeros of Chebyshev polynomial R_M defined in (1). Substituting each node x_k into (2), it can be expressed in the matrix form as

$$\begin{bmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ u(x_M) \end{bmatrix} = \begin{bmatrix} R_0(x_1) & R_1(x_1) & \cdots & R_{M-1}(x_1) \\ R_0(x_2) & R_1(x_2) & \cdots & R_{M-1}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ R_0(x_M) & R_1(x_M) & \cdots & R_{M-1}(x_M) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{M-1} \end{bmatrix},$$

which is denoted by $\mathbf{u} = \mathbf{Rc}$. Since $\mathbf{R}$ is invertible by Lemma 2.2 (iii), $\mathbf{c} = \mathbf{R}^{-1}\mathbf{u}$. Next, consider the single layer integration of u from a to x_k denoted by $U^{(1)}$, we obtain

$$U^{(1)}(x_k) = \int_a^{x_k} u(\xi)d\xi = \sum_{n=0}^{M-1} c_n \int_a^{x_k} R_n(\xi)d\xi = \sum_{n=0}^{M-1} c_n \bar{R}_n(x_k)$$

for $k \in \{1, 2, 3, \dots, M\}$ or in the matrix form:

$$\begin{bmatrix} U^{(1)}(x_1) \\ U^{(1)}(x_2) \\ \vdots \\ U^{(1)}(x_M) \end{bmatrix} = \begin{bmatrix} \bar{R}_0(x_1) & \bar{R}_1(x_1) & \cdots & \bar{R}_{M-1}(x_1) \\ \bar{R}_0(x_2) & \bar{R}_1(x_2) & \cdots & \bar{R}_{M-1}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ \bar{R}_0(x_M) & \bar{R}_1(x_M) & \cdots & \bar{R}_{M-1}(x_M) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{M-1} \end{bmatrix},$$

which we denote it by $\mathbf{U}^{(1)} = \bar{\mathbf{R}}\mathbf{c} = \bar{\mathbf{R}}\mathbf{R}^{-1}\mathbf{u} \coloneqq \mathbf{Au}$, where $\mathbf{A} = \bar{\mathbf{R}}\mathbf{R}^{-1} \coloneqq [a_{ki}]_{M\times M}$ is called the "Chebyshev integration matrix" for the developed FIM-CBS, i.e.,

$$U^{(1)}(x_k) = \int_a^{x_k} u(\xi)d\xi = \sum_{i=1}^{M} a_{ki} u(x_i).$$

Next, consider the double-layer integration of u from a to x_k denoted by $U^{(2)}$, we have

$$U^{(2)}(x_k) = \int_a^{x_k} \int_a^{\xi_2} u(\xi_1)d\xi_1 d\xi_2 = \sum_{i=1}^{M} a_{ki} \int_a^{x_i} u(\xi_1)d\xi_1 = \sum_{i=1}^{M}\sum_{j=1}^{M} a_{ki}a_{ij}u(x_j)$$

for $k \in \{1, 2, 3, \dots, M\}$, it can be written in the matrix form as $\mathbf{U}^{(2)} = \mathbf{A}^2\mathbf{u}$. Similarly, for the m-layer integration of u from a to x_k denoted by $U^{(m)}$ can be expressed as

$$U^{(m)}(x_k) = \int_a^{x_k} \dots \int_a^{\xi_2} u(\xi_1)d\xi_1 \dots d\xi_m = \sum_{i_m=1}^{M} \dots \sum_{j=1}^{M} a_{ki_m} \dots a_{i_1 j} u(x_j)$$

for $k \in \{1, 2, 3, \dots, M\}$, whose equation can be composed as $\mathbf{U}^{(m)} = \mathbf{A}^m\mathbf{u}$.

3. Numerical Algorithm for Solving Time-Fractional BBMB Equations

Before embarking on the details of constructing a numerical algorithm via the developed FIM-CBS for solving time-fractional BBMB equations, we provide the basic definitions of fractional derivatives, the necessary notations and some important facts. More details on basic results of fractional calculus can be found in Podlubny (1999).

Definition 3.1. (Podlubny, 1999) A real-valued function $u(t)$, $t > 0$ can be defined on the space C_μ, $\mu \in \mathbb{R}$, if there exist a real number $\rho > \mu$ such that $u(t) = t^\rho u_1(t)$, where $u_1(t) \in C[0, \infty)$ and it is defined on the space C_μ^n, if and only if $u^{(n)} \in C_\mu$, $n \in \mathbb{N}$.

Definition 3.2. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of an integrable function $u \in C_\mu$, $\mu > -1$ is defined by

$$I^\alpha u(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(s)}{(t-s)^{1-\alpha}} ds & \text{for } \alpha > 0, \\ u(t) & \text{for } \alpha = 0, \end{cases}$$

where $\Gamma(\cdot)$ is the well-known Gamma function.

Definition 3.3. The Caputo fractional-order derivative of $u \in C_{-1}^m$, $m \in \mathbb{N}$, is defined by

$$D^{\alpha}u(t) = I^{m-\alpha}D^{m}u(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{u^{(m)}(s)}{(t-s)^{1-m+\alpha}}ds & \text{for } \alpha \in (m-1, m), \\ u^{(m)}(t) & \text{for } \alpha = m. \end{cases}$$

Next, the time-fractional BBMB equations in one-dimensional space considered by Kumar and Kumar (2014) can be written as

$$D_t^{\alpha}u - \frac{\partial^3 u}{\partial x^2 \partial t} + \frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} = f(x,t), \;\; x \in (0,L), \;\; t \in (0,T],$$

subject to the initial condition

$$u(x,0) = \phi(x), \;\; x \in [0,L], \tag{3}$$

and the Dirichlet boundary conditions

$$u(0,t) = \psi_1(t), \;\; u(L,t) = \psi_2(t), \;\; t \in (0,T], \tag{4}$$

where L and T are positive real numbers, $f(x,t)$, $\phi(x)$, $\psi_1(t)$ and $\psi_2(t)$ are the given smooth functions and $\alpha \in (0,1]$ is the orders of time fractional derivative term.

Let us first use the technique of linearization to handle (3) by taking the iteration at time $t_m = m(\Delta t)$, where Δt is a time step and $m \in \mathbb{N}$. Then, we obtain

$$D_t^{\alpha}u|_{t=t_m} - \frac{\partial^3 u}{\partial x^2 \partial t}\Big|_{t=t_m} + \frac{\partial u^m}{\partial x} + u^{m-1}\frac{\partial u^m}{\partial x} = f(x,t_m). \tag{5}$$

where $u^m = u(x,t_m)$ is the numerical solution at m^{th} iteration. Next, consider the fractional-order derivative term with respect to time in the Caputo sense, we have

$$D_t^{\alpha}u|_{t=t_m} = \frac{1}{\Gamma(1-\alpha)}\int_0^{t_m}\frac{u_s(x,s)}{(t_m-s)^{\alpha}}ds = \frac{1}{\Gamma(1-\alpha)}\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}}\frac{u_s(x,s)}{(t_m-s)^{\alpha}}ds. \tag{6}$$

We use the first order forward difference quotient to approximate the time derivative term in *(7)*. For convenience, we also let $j = m-i-1$. Then,

$$\begin{aligned} D_t^{\alpha}u|_{t=t_m} &\approx \frac{1}{\Gamma(1-\alpha)}\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}}(t_m-s)^{-\alpha}\left(\frac{u^{i+1}-u^i}{\Delta t}\right)ds \\ &= \frac{1}{\Gamma(1-\alpha)}\sum_{i=0}^{m-1}\left(\frac{u^{i+1}-u^i}{\Delta t}\right)\left(\frac{(t_m-t_i)^{1-\alpha}-(t_m-t_{i+1})^{1-\alpha}}{1-\alpha}\right) \\ &= \frac{(\Delta t)^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{i=0}^{m-1}\left(\frac{u^{i+1}-u^i}{\Delta t}\right)\left((m-i)^{1-\alpha}-(m-i-1)^{1-\alpha}\right) \\ &= \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{m-1}\left(u^{m-j}-u^{m-j-1}\right)\left((j+1)^{1-\alpha}-j^{1-\alpha}\right) \\ &= \sum_{j=0}^{m-1} w_j\left(u^{m-j}-u^{m-j-1}\right), \end{aligned} \tag{7}$$

where $w_j = \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}((j+1)^{1-\alpha} - j^{1-\alpha})$. Next, we consider the third derivative term with respect to twice spaces and single time in (6). We approximate the time derivative of this term by using the first order forward difference quotient, we have

$$\frac{\partial^3 u}{\partial x^2 \partial t}\Big|_{t=t_m} = \frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial t}\right)\Big|_{t=t_m} = \frac{\partial^2}{\partial x^2}\left(\frac{u^m-u^{m-1}}{\Delta t}\right) = \frac{1}{\Delta t}\left(\frac{\partial^2 u^m}{\partial x^2} - \frac{\partial^2 u^{m-1}}{\partial x^2}\right). \tag{8}$$

Thus, we replace (8) and (9) into (6) to obtain

$$\sum_{j=0}^{m-1} w_j\left(u^{m-j}-u^{m-j-1}\right) - \frac{1}{\Delta t}\left(\frac{\partial^2 u^m}{\partial x^2} - \frac{\partial^2 u^{m-1}}{\partial x^2}\right) + \frac{\partial u^m}{\partial x} + u^{m-1}\frac{\partial u^m}{\partial x} = f(x,t_m). \tag{9}$$

Next, to eliminate the derivative terms from the above equation, we apply the developed FIM-CBS in Section 2 by taking the twice layer integration. Then, we get the following equation at each zero of Chebyshev polynomial x_k for $k \in \{1,2,3,\dots,M\}$, we have

$$w_0 \int_0^{x_k} \int_0^{\eta} (u^m - u^{m-1}) d\xi d\eta + \sum_{j=1}^{m-1} w_j \int_0^{x_k} \int_0^{\eta} \left(u^{m-j} - u^{m-j-1}\right) d\xi d\eta - \frac{u^m}{\Delta t} + \frac{u^{m-1}}{\Delta t} + \int_0^{x_k} u^m d\eta + \int_0^{x_k} \int_0^{\eta} \left(u^{m-1} \frac{\partial u^m}{\partial \xi}\right) d\xi d\eta + d_1 x_k + d_2 = \int_0^{x_k} \int_0^{\eta} f(\xi, t_m) d\xi d\eta, \tag{10}$$

where d_1 and d_2 are arbitrary constants emerged in the process of integration.
Then, we consider the nonlinear term in (10). By using the integration by parts, we have

$$\begin{aligned} q(x_k) &:= \int_0^{x_k} \int_0^{\eta} \left(u^{m-1} \frac{\partial u^m}{\partial \xi}\right) d\xi d\eta \\ &= \int_0^{x_k} u^{m-1} u^m d\eta - \int_0^{x_k} \int_0^{\eta} \frac{\partial u^{m-1}}{\partial \xi} u^m d\xi d\eta \\ &= \int_0^{x_k} u^{m-1} u^m d\eta - \int_0^{x_k} \int_0^{\eta} \sum_{n=0}^{M-1} c_n^{m-1} R_n'(\xi)\, u^m d\xi d\eta \\ &= \int_0^{x_k} u^{m-1} u^m d\eta - \int_0^{x_k} \int_0^{\eta} \mathbf{R}'(\xi) \mathbf{c}^{m-1} u^m d\xi d\eta \\ &= \int_0^{x_k} u^{m-1} u^m d\eta - \int_0^{x_k} \int_0^{\eta} \mathbf{R}'(\xi) \mathbf{R}^{-1} \mathbf{u}^{m-1} u^m d\xi d\eta \end{aligned}$$

where $\mathbf{R}'(\xi) = \left[R_0'(\xi), R_1'(\xi), R_2'(\xi), \dots, R_{M-1}'(\xi)\right]$. Thus, when we vary each x_k in (11) for $k \in \{1, 2, 3, \dots, M\}$, (11) can be expressed in the matrix form as

$$\begin{bmatrix} q(x_1) \\ q(x_2) \\ \vdots \\ q(x_M) \end{bmatrix} = \mathbf{A} \begin{bmatrix} u^{m-1}(x_1) u^m(x_1) \\ u^{m-1}(x_2) u^m(x_2) \\ \vdots \\ u^{m-1}(x_M) u^m(x_M) \end{bmatrix} - \mathbf{A}^2 \begin{bmatrix} \mathbf{R}'(x_1) \mathbf{R}^{-1} \mathbf{u}^{m-1} u^m(x_1) \\ \mathbf{R}'(x_2) \mathbf{R}^{-1} \mathbf{u}^{m-1} u^m(x_2) \\ \vdots \\ \mathbf{R}'(x_M) \mathbf{R}^{-1} \mathbf{u}^{m-1} u^m(x_M) \end{bmatrix}. \tag{11}$$

For computational convenience, we reduce the above equation into the matrix form:

$$\mathbf{q} = \mathbf{A} \mathrm{diag}(\mathbf{u}^{m-1}) \mathbf{u}^m - \mathbf{A}^2 \mathrm{diag}\left(\mathbf{R}' \mathbf{R}^{-1} \mathbf{u}^{m-1}\right) \mathbf{u}^m. \tag{12}$$

Consequently, for $k \in \{1, 2, 3, \dots, M\}$ by consuming (12) and the developed FIM-CBS, we can convert (10) into the matrix form as follows

$$\begin{aligned} & w_0 \mathbf{A}^2 (\mathbf{u}^m - \mathbf{u}^{m-1}) + \sum_{j=1}^{m-1} w_j \mathbf{A}^2 \left(\mathbf{u}^{m-j} - \mathbf{u}^{m-j-1}\right) - \frac{1}{\Delta t} \mathbf{u}^m + \frac{1}{\Delta t} \mathbf{u}^{m-1} \\ & + \mathbf{A} \mathbf{u}^m + \mathbf{A} \mathrm{diag}(\mathbf{u}^{m-1}) \mathbf{u}^m - \mathbf{A}^2 \mathrm{diag}\left(\mathbf{R}' \mathbf{R}^{-1} \mathbf{u}^{m-1}\right) \mathbf{u}^m + d_1 \mathbf{x} + d_2 \mathbf{i} = \mathbf{A}^2 \mathbf{f} \end{aligned}$$

or it can be simplified as

$$\begin{aligned} & \left(w_0 \mathbf{A}^2 - \frac{1}{\Delta t} \mathbf{I} + \mathbf{A} + \mathbf{A} \mathrm{diag}(\mathbf{u}^{m-1}) - \mathbf{A}^2 \mathrm{diag}\left(\mathbf{R}' \mathbf{R}^{-1} \mathbf{u}^{m-1}\right)\right) \mathbf{u}^m + d_1 \mathbf{x} + d_2 \mathbf{i} \\ & \quad = \mathbf{A}^2 \mathbf{f} - \sum_{j=1}^{m-1} w_j \mathbf{A}^2 \left(\mathbf{u}^{m-j} - \mathbf{u}^{m-j-1}\right) + \left(w_0 \mathbf{A}^2 - \frac{1}{\Delta t} \mathbf{I}\right) \mathbf{u}^{m-1}, \end{aligned} \tag{13}$$

where $\mathbf{I}$ is the identity matrix, $\mathbf{x} = [x_1, x_2, x_3, \dots, x_M]^T$, $\mathbf{i} = [1, 1, 1, \dots, 1]^T$, $\mathbf{A} = \bar{\mathbf{R}} \mathbf{R}^{-1}$, $\mathbf{u}^m = [u(x_1, t_m), u(x_2, t_m), \dots, u(x_M, t_m)]^T$, $\mathbf{f} = [f(x_1, t_m), f(x_2, t_m), \dots, f(x_M, t_m)]^T$,

and
$$\mathbf{R}' = \begin{bmatrix} \mathbf{R}'(x_1) \\ \mathbf{R}'(x_2) \\ \vdots \\ \mathbf{R}'(x_M) \end{bmatrix} = \begin{bmatrix} R_0'(x_1) & R_1'(x_1) & \cdots & R_{M-1}'(x_1) \\ R_0'(x_2) & R_1'(x_2) & \cdots & R_{M-1}'(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ R_0'(x_M) & R_1'(x_M) & \cdots & R_{M-1}'(x_M) \end{bmatrix}.$$

From the given boundary conditions (5), we can transform them into the vector forms by using linear combination of Chebyshev polynomial (2) at m^{th} iteration as follows

$$u(0, t_m) = \sum_{n=0}^{M-1} c_n^m R_n(0) = \sum_{n=0}^{M-1} c_n^m (-1)^n = \mathbf{h}_l \mathbf{c}^m = \mathbf{h}_l \mathbf{R}^{-1} \mathbf{u}^m = \psi_1(t_m), \tag{14}$$

$$u(L, t_m) = \sum_{n=0}^{M-1} c_n^m R_n(L) = \sum_{n=0}^{M-1} c_n^m (1)^n = \mathbf{h}_r \mathbf{c}^m = \mathbf{h}_r \mathbf{R}^{-1} \mathbf{u}^m = \psi_1(t_m), \tag{15}$$

where $t_m = m(\Delta t)$ for $m \in \mathbb{N}$, $\mathbf{h}_l = [1, -1, 1, \dots, (-1)^{M-1}]$ and $\mathbf{h}_r = [1, 1, 1, \dots, 1]$. Finally, from (13), (14) and (15), we can construct the system of linear equation at the iteration t_m for $m \in \mathbb{N}$ which contains $M + 2$ unknowns as follows

$$\begin{bmatrix} \mathbf{K} & \mathbf{x} & \mathbf{i} \\ \mathbf{h}_l\mathbf{R}^{-1} & 0 & 0 \\ \mathbf{h}_r\mathbf{R}^{-1} & 0 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{u}^m \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A}^2\mathbf{f} - \mathbf{s} + (w_0\mathbf{A}^2 - \frac{1}{\Delta t}\mathbf{I})\mathbf{u}^{m-1} \\ \psi_1(t_m) \\ \psi_2(t_m) \end{bmatrix}, \tag{16}$$

where $\mathbf{K} = w_0\mathbf{A}^2 - \frac{1}{\Delta t}\mathbf{I} + \mathbf{A} + \mathbf{A}\text{diag}(\mathbf{u}^{m-1}) - \mathbf{A}^2\text{diag}\left(\mathbf{R}'\mathbf{R}^{-1}\mathbf{u}^{m-1}\right)$, $\mathbf{s} = \mathbf{0}$ for $m = 1$ and $\mathbf{s} = \sum_{j=1}^{m-1} w_j\mathbf{A}^2\left(\mathbf{u}^{m-j} - \mathbf{u}^{m-j-1}\right)$ for $m > 1$. Thus, the solution $\mathbf{u}^m$ can be found by solving the system (16) with starting from $\mathbf{u}^0 = [\phi(x_1), \phi(x_2), \phi(x_3), \dots, \phi(x_M)]^T$. We notice here that for the terminal time T, the numerical solution $u(x, T)$ for each arbitrary $x \in (0, L)$ can be computed from

$$u(x,T) = \sum_{n=0}^{M-1} c_n R_n(x) = \mathbf{R}(x)\mathbf{c} = \mathbf{R}(x)\mathbf{R}^{-1}\mathbf{u}^m,$$

where $\mathbf{R}(x) = [R_0(x), R_1(x), R_2(x), \dots, R_{M-1}(x)]$ and $\mathbf{u}^m$ is the final m^{th} iteration of (16).

Algorithm. The Numerical Algorithm for Solving Time-Fractional BBMB Equations

Input: $\alpha, x, L, T, M, \Delta t, \phi(x), \psi_1(t), \psi_2(t)$ and $f(x,t)$.
Output: An approximate solution $u(x,T)$.
1: Set $x_k = \frac{L}{2}\left(\cos\left(\frac{2k-1}{2M}\pi\right) + 1\right)$ for $k \in \{1, 2, 3, \dots, M\}$.
2: Compute $\mathbf{x}$, $\mathbf{i}$, $\mathbf{h}_l$, $\mathbf{h}_r$, $\mathbf{I}$, $\mathbf{A}$, $\mathbf{R}$, $\bar{\mathbf{R}}$, $\mathbf{R}'$, $\mathbf{R}^{-1}$, $\mathbf{R}(x)$ and w_0.
3: Construct $\mathbf{u}^0 = [\phi(x_1), \phi(x_2), \phi(x_3), \dots, \phi(x_M)]^T$.
4: Calculate the total number of iterations $N = \frac{T}{\Delta t}$.
5: **for** $m = 1$ to N **do**
6: Set $t_m = m(\Delta t)$.
7: Set $\mathbf{s} = \mathbf{0}$.
8: **for** $j = 1$ to $m - 1$ **do**
9: Compute $w_j = \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}\left((j+1)^{1-\alpha} - j^{1-\alpha}\right)$.
10: Compute $\mathbf{s} = \mathbf{s} + w_j\mathbf{A}^2\left(\mathbf{u}^{m-j} - \mathbf{u}^{m-j-1}\right)$.
11: **end for**
12: Compute $\mathbf{K} = w_0\mathbf{A}^2 - \frac{1}{\Delta t}\mathbf{I} + \mathbf{A} + \mathbf{A}\text{diag}(\mathbf{u}^{m-1}) - \mathbf{A}^2\text{diag}\left(\mathbf{R}'\mathbf{R}^{-1}\mathbf{u}^{m-1}\right)$.
13: Compute $\mathbf{f} = [f(x_1, t_m), f(x_2, t_m), f(x_3, t_m) \dots, f(x_M, t_m)]^T$.
14: Find $\mathbf{u}^m$ by solving the iterative linear system (16).
15: **end for**
16: **return** $u(x,T) = \mathbf{R}(x)\mathbf{R}^{-1}\mathbf{u}^m$.

4. Numerical Experiments

In this section, we have applied the proposed algorithm based on the developed FIM-CBS for seeking numerical solutions of the time-fractional BBMB equations in order to demonstrate the efficiency and effectiveness of our scheme through several numerical examples which measured the accurate results by the average absolute error AAE $= |u^*(x,t) - u(x,t)|$, where u^* and u are the analytical and numerical solutions.

Example 1. Consider the time-fractional BBMB equation (3) with the source term

$$f(x,t) = \frac{3\sqrt{\pi}x^4(x-1)t^{\frac{3}{2}-\alpha}}{4\Gamma\left(\frac{5}{2}-\alpha\right)} + x^2t^{\frac{1}{2}}\left(5x^7t^{\frac{5}{2}} - 9x^6t^{\frac{5}{2}} + 4x^5t^{\frac{5}{2}} + 5x^2t - 4xt - 30x + 18\right)$$

subject to the initial condition

$$u(x,0) = 0, \quad x \in [0,1], \tag{17}$$

and the Dirichlet boundary conditions

$$u(0,t) = 0, \quad u(1,t) = 0, \quad t \in (0,1]. \tag{18}$$

The analytical solution given by Shen and Zhu (2018) is $u^*(x,t) = x^4(x-1)t^{\frac{3}{2}}$. In the numerical testing, we compare the approximate results obtained by our algorithm with a Crank-Nicolson linear difference scheme (CNLDS) proposed by Shen and Zhu (2018) which measured by AAEs for $\alpha = 0.5$ and $\Delta t = 0.001$ at various $M \in \{10, 20, 40, 80\}$ as shown in Table 1. We can see that our algorithm gives higher accuracy than the CNLDS under the same parameters and conditions. Moreover, we also illustrate the AAEs at the final time $T = 1$ for $\alpha = 0.5$, $M = 40$ and the different time steps $\Delta t \in \{0.05, 0.01, 0.005, 0.001\}$ in Table 2 and for $\Delta t = 0.001$, $M = 40$ and the various fractional orders of derivative $\alpha \in \{0.1, 0.3, 0.7, 0.9\}$ in Table 3. Finally, the graph of our numerical solutions $u(x,t)$ at the different times t and the surface plot of our numerical solutions $u(x,t)$ whole domains are provided in Figure 1.

Table 1. AAEs for $\alpha = 0.5$, $\Delta t = 0.001$ and $M \in \{10, 20, 40, 80\}$ of Example 1

M	CNLDS	Our algorithm
10	1.841489×10^{-3}	1.7810×10^{-5}
20	5.047981×10^{-4}	1.7763×10^{-5}
40	1.321991×10^{-4}	1.7751×10^{-5}
80	3.365747×10^{-5}	1.7748×10^{-5}

Example 2. Consider the time-fractional BBMB equation (3) with the source term as

$$f(x,t) = \frac{2e^x t^{2-\alpha}}{\Gamma(3-\alpha)} + te^x(t^3 + t - 2)$$

subject to the initial condition (17) and the Dirichlet boundary conditions

$$u(0,t) = t^2, \;\; u(1,t) = et^2, \;\;\; t \in (0,1].$$

The analytical solution given by Esen and Tasbozan (2015) is $u^*(x,t) = t^2 e^x$. For the numerical examination, we choose the parameters $\alpha = 0.5$, $M = 40$ and the many time steps $\Delta t \in \{0.05, 0.01, 0.005, 0.001\}$ in order to show AAEs in Table 4.

Also, we vary the fractional orders of derivative $\alpha \in \{0.1, 0.3, 0.7, 0.9\}$ at $\Delta t = 0.001$ and $M = 40$ to display the AAEs in Table 5. Figure 2 provides the graphical solutions of this problem including the graph of our numerical solutions $u(x,t)$ at the different times t and the surface plot of our numerical solutions $u(x,t)$.

Example 3. Consider the time-fractional BBMB equation (3) with the source term as

$$f(x,t) = \frac{t^{1-\alpha}\sin(\pi x)}{\Gamma(2-\alpha)} + \pi^2 \sin(\pi x) + \pi t \sin(\pi x) + \frac{\pi t^2}{2}\sin(2\pi x)$$

subject to the same initial condition (17) and the Dirichlet boundary conditions (18). The analytical solution given by Zarebnia and Parvaz (2017) is $u^*(x,t) = t\sin(\pi x)$. In the numerical testing of this problem, we consider AAEs of this problem by selecting the same parameters in Example 2 which varies along the time steps Δt and the fractional orders of derivative α as shown in Tables 6 and 7, respectively. Finally, we demonstrate the plotting numerical solutions $u(x,t)$ at the different times t and the surface plot of $u(x,t)$ in Figure 3.

Table 2. AAEs at time $T = 1$ for $\alpha = 0.5$, $M = 40$ and various Δt of Example 1

x	$\Delta t = 0.05$	$\Delta t = 0.01$	$\Delta t = 0.005$	$\Delta t = 0.001$
0.2	1.3072×10^{-3}	2.7980×10^{-5}	1.4266×10^{-5}	2.9342×10^{-6}
0.4	1.8644×10^{-3}	3.8426×10^{-5}	1.9242×10^{-5}	3.8423×10^{-6}
0.6	1.3842×10^{-3}	2.9035×10^{-4}	1.4669×10^{-4}	2.9731×10^{-5}
0.8	2.5887×10^{-3}	5.4398×10^{-4}	2.7512×10^{-4}	5.5861×10^{-5}

Table 3. AAEs at time $T = 1$ for $\Delta t = 0.001$, $M = 40$ and various α of Example 1

x	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.7$	$\alpha = 0.9$
0.2	2.8624×10^{-6}	2.9140×10^{-6}	2.8699×10^{-6}	2.4680×10^{-6}
0.4	4.0394×10^{-6}	3.9104×10^{-6}	3.9486×10^{-6}	4.7687×10^{-6}
0.6	3.0072×10^{-5}	2.9863×10^{-5}	2.9838×10^{-5}	3.0944×10^{-5}
0.8	5.6196×10^{-5}	5.5998×10^{-5}	5.5923×10^{-5}	5.6838×10^{-5}

Table 4. AAEs at time $T = 1$ for $\alpha = 0.5$, $M = 40$ and various Δt of Example 2

x	$\Delta t = 0.05$	$\Delta t = 0.01$	$\Delta t = 0.005$	$\Delta t = 0.001$
0.2	1.4907×10^{-4}	6.4013×10^{-5}	4.1055×10^{-5}	1.0379×10^{-5}
0.4	5.1386×10^{-4}	6.1063×10^{-5}	4.6138×10^{-5}	1.2941×10^{-5}
0.6	9.1238×10^{-4}	1.1182×10^{-5}	2.3799×10^{-5}	9.0593×10^{-6}
0.8	9.7423×10^{-4}	3.9815×10^{-5}	5.5279×10^{-6}	2.2631×10^{-6}

Table 5. AAEs at time $T = 1$ for $\Delta t = 0.001$, $M = 40$ and various α of Example 2

x	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.7$	$\alpha = 0.9$
0.2	1.1518×10^{-5}	1.1292×10^{-5}	4.5130×10^{-6}	2.8758×10^{-5}
0.4	1.4991×10^{-5}	1.4551×10^{-5}	2.9661×10^{-6}	5.3208×10^{-5}
0.6	1.1558×10^{-5}	1.1002×10^{-5}	2.4827×10^{-6}	6.6751×10^{-5}
0.8	4.3173×10^{-6}	3.8567×10^{-6}	6.8200×10^{-6}	5.6685×10^{-5}

Table 6. AAEs at time $T = 1$ for $\alpha = 0.5$, $M = 40$ and various Δt of Example 3

x	$\Delta t = 0.05$	$\Delta t = 0.01$	$\Delta t = 0.005$	$\Delta t = 0.001$
0.2	1.0218×10^{-3}	1.9609×10^{-4}	9.7531×10^{-5}	1.9424×10^{-5}
0.4	7.5199×10^{-4}	1.4293×10^{-4}	7.1000×10^{-5}	1.4126×10^{-5}
0.6	4.4809×10^{-4}	8.8329×10^{-5}	4.4082×10^{-5}	8.8029×10^{-6}
0.8	9.2119×10^{-4}	1.7832×10^{-4}	8.8787×10^{-5}	1.7698×10^{-5}

Table 7. AAEs at time $T = 1$ for $\Delta t = 0.001$, $M = 40$ and various α of Example 3

x	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.7$	$\alpha = 0.9$
0.2	1.9561×10^{-5}	1.9499×10^{-5}	1.9335×10^{-5}	1.9232×10^{-5}
0.4	1.4256×10^{-5}	1.4198×10^{-5}	1.4039×10^{-5}	1.3936×10^{-5}
0.6	8.7989×10^{-6}	8.7996×10^{-6}	8.8102×10^{-6}	8.8226×10^{-6}
0.8	1.7765×10^{-5}	1.7734×10^{-5}	1.7658×10^{-5}	1.7615×10^{-5}

(a) The solutions at different time t (b) The surface plot of solutions

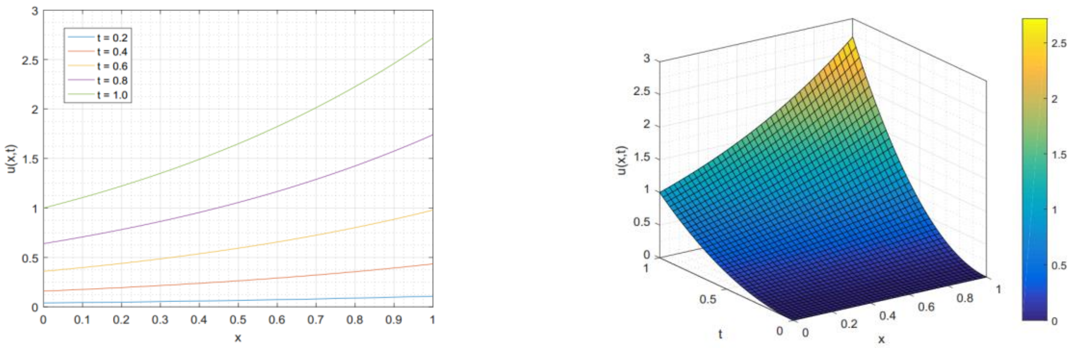

Figure 1. Graphical solutions for $\alpha = 0.5$, $M = 40$ and $\Delta t = 0.001$ of Example 1

(a) The solutions at different time t (b) The surface plot of solutions

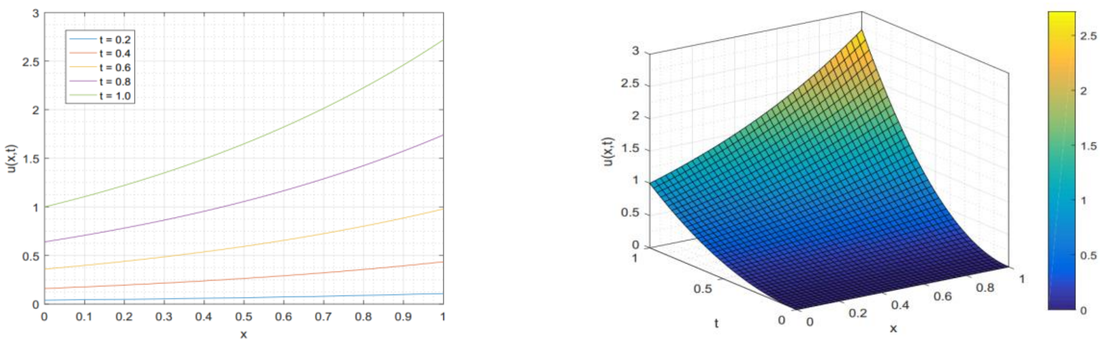

Figure 2. Graphical solutions for $\alpha = 0.5$, $M = 40$ and $\Delta t = 0.001$ of Example 2

(a) The solutions at different time t (b) The surface plot of solutions

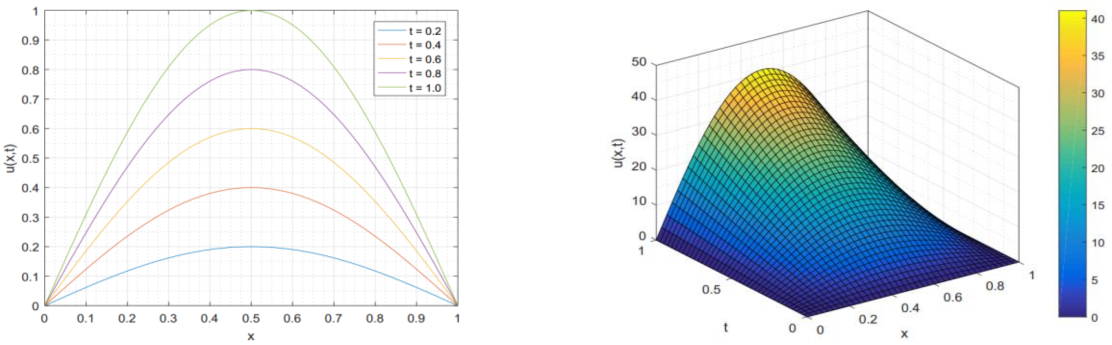

Figure 3. Graphical solutions for $\alpha = 0.5$, $M = 40$ and $\Delta t = 0.001$ of Example 3

5. Conclusions

In this paper, we devise the accurate and efficient numerical algorithm based on the developed FIM-CBS in the general range for finding the approximate solutions of the time-fractional BBMB equations. The fractional order derivative is in the sense of Caputo. The numerical experiments demonstrate that our method produces a much higher accuracy than the CNLDS under the same parameters and conditions for varying the numbers of discretization (see Example 1). We notice that it provides more accuracy even when we use a small number of nodal points M. Evidently, when we decrease the time step Δt, it furnishes more accurate results. In addition, our numerical algorithm gives a good performance on the fractional orders derivative $\alpha \in (0,1)$ and actually it can be easily applied to other nonlinear fractional PDEs. An interesting direction for our future work is to extend our technique to solve the space-fractional BBMB equations and other nonlinear fractional PDEs.

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