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Original Article

Information entropic measures for a trigonometric inversely quadratic plus Coulombic Hyperbolic Potential

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Abstract

In this study, the analytical solution of the Schrödinger equation for the Trigonometric Inversely Quadratic plus Coulombic Hyperbolic Potential via the methodology of the supersymmetric approach was obtained. The energy equation and its corresponding wave functions were fully calculated. The theoretic quantities such as Shannon entropy and Fisher information were calculated using the normalized radial wave function. The results obtained for Shannon entropy satisfied Beckner, Bialynicki-Birula and Mycieslki (BBM) principle and Cramer Rao uncertainty inequality for Fisher information. These results are in excellent agreement with those in the literature. The result of our study goes against the observation pointed out by Okon *et al.* in their recent paper, who claimed that information entropic measures cannot be studied under Trigonometric Inversely Quadratic plus Coulombic Hyperbolic Potential.

Keywords: eigensolutions, wave equations, bound states, fisher information, Shannon entropy

1. Introduction

The understanding of quantum mechanical systems is usually provided by the theoretic quantities such as entropic measures and Fisher information. These can be seen in the study of the electronic structure of atoms and molecules, characterization of complex signals of quantum mechanical systems in various areas of science (Howard, Sen, Borgoo & Geerlings, 2009; Ikhdair & Sever, 2008; Lopez-Ruiz, Nagy, Romera & Sanudo, 2009; Manzano, Yáñez, & Dehesa, 2010). The entropic uncertainty relations serve as alternatives to Heisenberg uncertainty relation and these have been tested by different authors (Angulo, Antolin, Zarzo & Cuchi, 1999; Bialynicki-Birula & Mycielski, 1975; Dehesa, Laguna & Sagar, 2010; Dehesa, Yáñez, Aptekarev, & Buyarov, 1998; Galindo & Pascual, 1978; Martínez-Finkelshtein, & Sorokin, 2006; Orlowski, 1997; Sánchez-Ruiz, 1997; Shannon, 1948). Quantum information theory has a direct relationship with the Heisenberg uncertainty principle, which plays a very significant role in the simultaneous measurement of position and momentum of quantum mechanical particles. This entropic uncertainty relation relates to the position and momentum spaces obtained by Beckner, Bialynicki-Birula, and Mycieslki (BBM) (Lopez-Ruiz *et al.*, 2009) is given by

$$S(\rho)+S(\gamma)\geq D(1+\log\pi), \tag{1}$$

where D denotes the spatial dimension. In 1948, a new uncertainty relation, based on Shannon entropy, was established as a basic tool for investigating the fundamental limit of signal processing (Shannon, 1948). Recently, the theoretic quantities have been studied for different models using different physical potential terms due to its wider applications. For instance, Donga, Sunb, Dong, and Draayer (2014), studied Quantum information entropies for a squared tangent potential well, their result was seen to obey

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Bialynicki-Birula, Mycielski inequality, Onate, Onyeaju, Ikot, Ebomwonyi, and Idiodi (2019), studied Fisher information and uncertainty relations for potential family. These authors deduced some uncertainty relations for Fisher information. Yahya, Oyewumi, and Sen (2014), in one of their papers, studied Information and complexity measures for the ring-shaped modified Kratzer potential. Using Laguerre polynomials, Gegenbauer polynomials, and spherical harmonics, these authors calculated the Tsallis and Renyi entropies for $q = 2$. Najafizade, Hassanabadi, and Zarrinkamar (2016), investigated nonrelativistic Shannon information entropy for Kratzer potential. The result obtained satisfied BBM inequality. Sun, Dong, and Saad, (2013) calculated the position and momentum space information entropies using Asymmetric–trigonometric Rosen-Morse potential. Okon, Isonguyo, Antia, Ikot, and Popoola (2020a), studied Fisher and Shannon information entropies for a non-central inversely quadratic plus exponential Mie-type potential. These authors calculated expectation values and determined the Heisenberg uncertainty relation. Of all these studies and other works not captured in this paper, Okon, Akaninyene, Akaninyene, and Imeh (2020b) clearly stated that it has been a difficult task for authors to apply Trigonometric Inversely Quadratic plus Coulombic Hyperbolic Potential (TIQPCHP) to the study the information entropic measures. The authors claimed that the potential does not belong to the Pöschl-Teller potential due to its combination, hence, they claimed the impossibility to obtain the theoretic quantities under the mentioned potential. They also pointed out that the potential is applicable only for a physical system where the bound state energies obtained can be used to study the motion of quarks, mesons, neutrinos and other elementary particles in high energy physics. Based on this argument, the present study wants to examine the Fisher information and Shannon entropy for the potential under consideration. The Trigonometric Inversely Quadratic plus Coulombic Hyperbolic Potential as given by Okon *et al.*, (2020b) reads

$$V(r) = \frac{v_0 sin\alpha}{r^2} + \frac{Acosh\alpha}{r} + B, \tag{2}$$

where V_0, A and B are constant and α is the screening parameter that characterizes the range of the potential. This potential can be reduced to another useful potential by giving numerical values to some of the potential parameters.

2. Bound State Solutions

For any physical quantum system, the original Schrödinger equation is given by Landau and Lifshitz (1977) and Schiff (1968).

$$\frac{p^2}{2\mu}\psi_{n\ell m}(r) = \left[E_{n\ell} - V(r)\right]\psi_{n\ell m}(r). \tag{3}$$

Setting the wave function $\psi_{n\ell m}(r) = \frac{R_{n,\ell}(r)Y_{\ell,m}(\theta,\phi)}{r}$, the radial part of the Schrödinger equation is given as

$$\frac{d^2R_{n,\ell}(r)}{dr^2} = \frac{2\mu}{\hbar^2}\left[V(r) - E_{n,\ell} + \frac{\ell(\ell+1)\hbar^2}{2\mu r^2}\right]R_{n,\ell}(r), \tag{4}$$

where $E_{n,\ell}$ is the non-relativistic energy, $V(r)$ is the interacting potential, μ is the reduced mass, $\hbar$ is the reduced Planck's constant, ℓ is the angular momentum quantum number, n is the quantum number and $R_{n,\ell}(r)$ is the wave function. The centrifugal term $\frac{\ell(\ell+1)}{r^2}$ in Equation (6) can be approximated using Greene-Aldrich approximation scheme (Greene & Aldrich, 1976).

$$\frac{\ell(\ell+1)}{r^2} \approx \frac{\ell(\ell+1)\delta^2}{(1-e^{-\delta r})^2}. \tag{5}$$

This approximation scheme is valid for α << 1. Substituting Equation (2) and Equation (5) into Equation (4), the radial Schrödinger equation becomes

$$\frac{d^2R_{n,\ell}(r)}{dr^2} = \left[\frac{V_{T_1}e^{-\delta r}}{(1-e^{-\delta r})^2} + \frac{V_{T_2}e^{-\delta r}}{1-e^{-\delta r}} + V_{T_3}\right]R_{n,\ell}(r), \tag{6}$$

where the following are used for mathematical simplicity.

$$V_{T_1} = \ell(\ell+1)\delta^2 + \frac{2\mu v_0 \delta^2 sin\alpha}{\hbar^2}, \tag{7}$$

$$V_{T_2} = 2\ell(\ell+1)\delta^2 + \frac{2\mu\delta(2v_0\delta sin\alpha + Acosh\alpha)}{\hbar^2}, \tag{8}$$

$$V_{T_3} = \ell(\ell+1)\delta^2 + \frac{2\mu\delta(v_0\delta sin\alpha + Acosh\alpha)}{\hbar^2} + \frac{2\mu(B - E_{n,\ell})}{\hbar^2}. \tag{9}$$

At this juncture, the methodology of supersymmetric quantum mechanics will be applied to obtain the energy equation of the radial Schrödinger equation. Thus, the ground state wave function for this system is written as

$$R_{0,\ell}(r) = exp\left(-\int W(r)dr\right), \tag{10}$$

where $W(r)$ in Equation (10) is called a superpotential function in supersymmetry quantum mechanics which gives a solution to the differential equation given in Equation (6). Substituting Equation (10) into Equation (6) leads to a Riccati equation of the form

$$W^2(r) - \frac{dW(r)}{dr} = \frac{V_{T_1}e^{-\delta r}}{(1-e^{-\delta r})^2} + \frac{V_{T_2}e^{-\delta r}}{1-e^{-\delta r}} + V_{T_3}. \tag{11}$$

For effective and accurate validity of a solution to Equation (6) and to validate the properties of both the left hand side and right hand side, a superpotential function of the form

$$W(r) = \rho_0 + \frac{\rho_1 e^{-\delta r}}{1-e^{-\delta r}}, \tag{12}$$

is proposed where ρ_0 and ρ_1 are two different constants that will soon be determined. In this work, it should be noted that, the boundary conditions of the wave functions should be satisfied, i.e. $\psi_{n,\ell}(r)/r$ becomes zero when r is infinite, and $\psi_{n,\ell}(r)/r$ is finite when r goes to zero. Obviously, it is only when $r \to \infty$, $\psi_{n,\ell}(r)$ is finite and $\psi_{n,\ell}(r) = 0$ at the origin point r = 0, the radial wave function can satisfy the boundary conditions. To make these regularity conditions, it requires that $\rho_1 > 0$, $\rho_0 > 0$ and $\rho_0 > \rho_1$. The verification of these will soon be shown. Substituting Equation (12) into Equation (11), the two superpotential constants in Equation (12) can be determine as follows

$$\rho_0^2 = \ell(\ell+1)\delta^2 + \frac{2\mu\delta(v_0\delta sin\alpha + Acosh\alpha)}{\hbar^2} + \frac{2\mu(B - E_{n,\ell})}{\hbar^2}, \tag{13}$$

$$\rho_1 = \delta\left(1 \pm \sqrt{(1+2\ell)^2 + \frac{8\mu v_0 sin\alpha}{\hbar^2}}\right), \tag{14}$$

$$\rho_0 = \frac{2\ell(\ell+1)\delta^2 + \frac{2\mu\delta(2v_0\delta sin\alpha + Acosh\alpha)}{\hbar^2} + \rho_1^2}{2\rho_1}. \tag{15}$$

The determination of the energy equation requires a partner potential constructed from the superpotential function. Thus, using Equation (12), the supersymmetric partner potentials in the form $V_\pm(r) = W^2(r) \pm \frac{dW(r)}{dr}$, can be written as follows:

$$V_+(r) = \rho_0^2 + \frac{\rho_1(2\rho_0 - \rho_1)e^{-\delta r}}{1-e^{-\delta r}} + \frac{\rho_1(\rho_1 - \alpha)e^{-\delta r}}{(1-e^{-\delta r})^2}, \tag{16}$$

$$V_-(r) = \rho_0^2 + \frac{\rho_1(2\rho_0 - \rho_1)e^{-\delta r}}{1-e^{-\delta r}} + \frac{\rho_1(\rho_1 + \alpha)e^{-\delta r}}{(1-e^{-\delta r})^2}. \tag{17}$$

Equation (16) and Equation (17) satisfied the shape invariance condition via mapping of the form $\rho_1 \to \rho_1 + \alpha$. Now, putting $\rho_1 = a_0$, a relationship can established between the partner potentials and a residual term $R(a_1)$ that is independent of the variable *r*. Thus,

$$V_+(a_0, r) = V_-(a_1, r) + R(a_1). \tag{18}$$

The a_0 is an old set of parameters while a_1 is a new set of parameters uniquely determined from a_0 via $a_1 = f(a_0) = a_a + \delta$. The relationship between the two sets of parameters occurs in a recurrence form as: $a_2 = a_0 + 2\delta$, $a_3 = a_0 + 3\delta$ and subsequently $a_n = a_0 + n\delta$. On this note, Equation (18) can be written in a recurrence relation of the forms

$$R(a_1) = \left[\rho_0^2 + \left(\frac{\left[2\rho_0^2 - \frac{4\mu B}{\hbar^2} + \frac{4\mu E_{n.\ell}}{\hbar^2}\right] + a_0^2}{2a_0}\right)^2\right] - \left[\rho_0^2 + \left(\frac{\left[2\rho_0^2 - \frac{4\mu B}{\hbar^2} + \frac{4\mu E_{n.\ell}}{\hbar^2}\right] + a_1^2}{2a_1}\right)^2\right], \tag{19}$$

$$R(a_2) = \left[\rho_0^2 + \left(\frac{\left[2\rho_0^2 - \frac{4\mu B}{\hbar^2} + \frac{4\mu E_{n.\ell}}{\hbar^2}\right] + a_1^2}{2a_1}\right)^2\right] - \left[\rho_0^2 + \left(\frac{\left[2\rho_0^2 - \frac{4\mu B}{\hbar^2} + \frac{4\mu E_{n.\ell}}{\hbar^2}\right] + a_2^2}{2a_2}\right)^2\right], \tag{20}$$

$$R(a_3) = \left[\rho_0^2 + \left(\frac{\left[2\rho_0^2 - \frac{4\mu B}{\hbar^2} + \frac{4\mu E_{n.\ell}}{\hbar^2}\right] + a_2^2}{2a_2}\right)^2\right] - \left[\rho_0^2 + \left(\frac{\left[2\rho_0^2 - \frac{4\mu B}{\hbar^2} + \frac{4\mu E_{n.\ell}}{\hbar^2}\right] + a_3^2}{2a_3}\right)^2\right], \tag{21}$$

$$R(a_4) = \left[\rho_0^2 + \left(\frac{\left[2\rho_0^2 - \frac{4\mu B}{\hbar^2} + \frac{4\mu E_{n.\ell}}{\hbar^2}\right] + a_3^2}{2a_3}\right)^2\right] - \left[\rho_0^2 + \left(\frac{\left[2\rho_0^2 - \frac{4\mu B}{\hbar^2} + \frac{4\mu E_{n.\ell}}{\hbar^2}\right] + a_4^2}{2a_4}\right)^2\right], \tag{22}$$

$$R(a_n) = \left[\rho_0^2 + \left(\frac{\left[2\rho_0^2 - \frac{4\mu B}{\hbar^2} + \frac{4\mu E_{n.\ell}}{\hbar^2}\right] + a_{n-1}^2}{2a_{n-1}}\right)^2\right] - \left[\rho_0^2 + \left(\frac{\left[2\rho_0^2 - \frac{4\mu B}{\hbar^2} + \frac{4\mu E_{n.\ell}}{\hbar^2}\right] + a_n^2}{2a_n}\right)^2\right]. \tag{23}$$

Following the formalism of shape invariance and supersymmetric quantum mechanics approach in conjunction with the negative partner potential, the full energy level of the system can be written as

$$E_{n,\ell} = \Phi - \frac{\delta^2\hbar^2}{2\mu}\left[\frac{\frac{4\mu v_0 sin\alpha}{\hbar^2} + \frac{2\mu A cosh\alpha)}{\delta\hbar^2} + 2\ell(\ell+1) + \left(n + \frac{1}{2} + \frac{1}{2}\sqrt{(1+2\ell)^2 + \frac{8\mu v_0 sin\alpha}{\hbar^2}}\right)^2}{1 + 2n + \sqrt{(1+2\ell)^2 + \frac{8\mu v_0 sin\alpha}{\hbar^2}}}\right]^2, \tag{24}$$

$$\Phi = \frac{\ell(\ell+1)\delta^2\hbar^2}{2\mu} + v_0\delta^2 sin\alpha + A\delta cosh\alpha + B. \tag{25}$$

The corresponding wave function is given by

$$R(y) = N_{n,\ell} y^{\bar{u}} (1-y)^{0.5(1+\bar{v})} P_n^{(2\bar{u},\bar{v})}(1-2y). \tag{26}$$

where the following are used for mathematical simplicity

$$\bar{u} = \sqrt{\frac{2\mu(B - E_{n,\ell} + v_0\delta^2 sin\alpha + A\delta cosh\alpha)}{\delta^2\hbar^2} + \ell(\ell+1)}, \tag{27}$$

$$\bar{v} = \sqrt{(1+2\ell)^2 + \frac{8\mu v_0 sin\alpha}{\hbar^2}}. \tag{28}$$

Detail of how the wave function in Equation (26) is obtained can be found in Appendix A.

2.1 Normalization constant

The parameter N in Equation (26) is a normalization factor which can easily be calculated using normalization condition. To calculate N, we recall the general condition.

$$\int_0^\infty R(r)^2 dr = 1. \tag{29}$$

With the transformation $y = e^{-\delta r}$ and $x = 1-2y,$ Equation (29) turns to

$$\frac{1}{2\delta}\int_1^{-1}\left(\frac{2}{1-x}\right)R(x)^2 dx = 1. \tag{30}$$

Substituting Equation (26) into Equation (30), leads to the following

$$\frac{N_{n,\ell}^2}{2\delta}\int_1^{-1}(1-x)^{2\bar{u}-1}(1+x)^{1+\bar{v}}\left[P_n^{(2\bar{u},\bar{v})}(x)\right]^2 dx = 1, \tag{31}$$

where a transformation of the form $1-x = 1-\left(\frac{1-x}{2}\right)$ is defined. Using the appropriate integral in the appendix, the normalization factor becomes

$$N_{n,\ell}^2 = -\frac{n!2\delta\bar{u}\Gamma(2\bar{u}+\bar{v}+n+1)}{\Gamma(2\bar{u}+1)\Gamma(\bar{v}+n+2)}. \tag{32}$$

3. Theoretic Quantities

In this section, some theoretic quantities such as Shannon entropy and Fisher information are calculated using probability density function. Fisher information is a quantity of an efficient measurement procedure used for estimating an ultimate quantum limits which regulates how well it is possible to determine the internal structure of a system. The quantity determines the uncertainty relations of a constant mass quantum system and measures the local homogeneity of a system. Fisher information is used to predict the localization of a particle in a system while Shannon entropy is used to determine the stability of a system.

3.1 Shannon entropy

Shannon entropy for position space and momentum space respectively are given as (Onate, Adebimpe, Adebesin, & Lukman 2018)

$$S(\rho) = -4\pi \int_0^\infty \rho(r) log \rho(r) dr, \tag{33}$$

$$S(\gamma) = -4\pi \int_0^\infty \gamma(p) log \gamma(p) dp. \tag{34}$$

Defining a transformation s = 1-y, Equation (33) becomes

$$S(\rho) = \frac{4\pi N_{n,\ell}^2 \Gamma(2\overline{u}+n+1)}{n!\Gamma(2\overline{u}+1)} \int_0^1 \aleph log \aleph ds, \tag{35}$$

where

$$\aleph = s^{2\overline{u}} (1-s)^{1+\overline{v}} {}_2F_1(-n, n+2\overline{v}+2\overline{u}+1; 2\overline{u}+1; s). \tag{36}$$

Using the appropriate integral given in the Appendix B, Equation (35) is simplified to have Shannon entropy for the position space as

$$S(\rho) = \frac{8\pi\overline{u}(n!)\Gamma(2\overline{u}+1)\Gamma(\overline{v}+n+3)\Gamma(2\overline{u}+\overline{v}+n+1)}{\Gamma(2\overline{u}+n)\Gamma(\overline{v}+n+2)\Gamma(2\overline{u}+\overline{v}+n+3)} log\left[(0.99)^{2\overline{u}}(0.01)^{1+\overline{v}} \frac{\Gamma(2\overline{u}+n+1)}{\Gamma(2\overline{u}+1)}\right]. \tag{37}$$

To obtain Shannon entropy for momentum space, we define x = -1+2y, and invoke it on Equation (34) to have

$$S(\gamma) = \frac{2\pi N_{n,\ell}^2}{\delta} \int_{-1}^1 \aleph_0 log \aleph_0 dx, \tag{38}$$

where

$$\aleph_0 = \left(\frac{1-x}{2}\right)^{2\overline{u}} \left(\frac{1+x}{2}\right)^{1+\overline{v}} \left[P_n^{(2\overline{u},\overline{v})}(x)\right]^2. \tag{39}$$

On correct substitutions using integral and formula in the Appendix B, the Shannon entropy for momentum space becomes

$$S(\gamma) = -\frac{4\pi\Gamma(2\overline{u}+n+1)\Gamma(2\overline{u}+\overline{v}+n+1)}{\Gamma(2\overline{u}+n)\Gamma(2\overline{u}+\overline{v}+n+2)} \times log\left[(0.99)^{2\overline{u}}(0.01)^{1+\overline{v}} \times \frac{[\Gamma(2\overline{u}+n+1)]^2}{n![\Gamma(2\overline{u}+1)]^2}\right]. \tag{40}$$

3.2 Fisher information

Fisher information for position space and momentum space respectively are given as Dehesa, Martinez-Finkelshtein, and SnchezRuiz (2001), Dehesa, Gonzalez-Ferez, and Sanchez-Moreno (2007), Romera, Sanchez-Moreno, and Dehesa (2005).

$$I(\rho) = \int_0^\infty \frac{1}{\rho(t)} \left[\frac{d\rho(t)}{dt}\right]^2 dt. \tag{41}$$

where

$$I(\gamma) = \int_0^\infty \frac{1}{\gamma(t)} \left[\frac{d\gamma(t)}{dt}\right]^2 dt. \tag{42}$$

Following the same procedure for Shannon entropic in the position space, the Fisher information for position space is obtain as

$$I(\rho) = N_{n,\ell}^2 \left[\overline{v}I_{\rho_1} + 4\overline{u}I_{\rho_2} + 4I_{\rho_3} + 2\overline{u}\overline{v}I_{\rho_4} + 2\overline{v}I_{\rho_5} + 4\overline{u}I_{\rho_6}\right], \tag{43}$$

where

$$\left.\begin{aligned}
&I_{\rho_1}=\int_0^1 s^{\bar{v}-2}(1-s)^{2\bar{u}}\,{}_2F_1(-n,n+2(\bar{u}+\bar{v});2\bar{u}+1;s)^2, I_{\rho_2}=\int_0^1 s^{\bar{v}}(1-s)^{2\bar{u}-2}\,{}_2F_1(-n,n+2(\bar{u}+\bar{v});2\bar{u}+1;s)^2,\\
&I_{\rho_3}=\int_0^1 s^{\bar{v}-2}(1-s)^{2\bar{u}}\,{}_2F_1(-n,n+2(\bar{u}+\bar{v});2\bar{u}+1;s)^2, I_{\rho_4}=\int_0^1 s^{\bar{v}-1}(1-s)^{2\bar{u}-1}\,{}_2F_1(-n,n+2(\bar{u}+\bar{v});2\bar{u}+1;s)^2,\\
&I_{\rho_5}=\int_0^1 s^{\bar{v}-1}(1-s)^{2\bar{u}}\,{}_2F_1(-n,n+2(\bar{u}+\bar{v});2\bar{u}+1;s)^2, I_{\rho_6}=\int_0^1 s^{\bar{v}-1}(1-s)^{2\bar{u}-1}\,{}_2F_1(-n,n+2(\bar{u}+\bar{v});2\bar{u}+1;s)^2.
\end{aligned}\right\} \tag{44}$$

Using the integral in the Appendix B, the complete Fisher information is given as

$$I(\rho)=(n!)^2\left[\begin{aligned}
&\frac{\bar{v}\Gamma(\bar{v}-1)^2\Gamma(2\bar{u}+n+2)\Gamma(2\bar{u}+\bar{v}+n+1)}{\Gamma(2\bar{u}+1)\Gamma(\bar{v}+n-1)\Gamma(\bar{v}+n+2)\Gamma(2\bar{u}+\bar{v}+n)}+\frac{4\bar{u}^2\Gamma(\bar{v}+1)\Gamma(2\bar{u}+n)}{(\bar{u}-1)\Gamma(\bar{v}+n+1)\Gamma(2\bar{u}+1)}\times\\
&\frac{\Gamma(2\bar{u}+\bar{v}+n+1)}{\Gamma(\bar{v}+n+2)\Gamma(2\bar{u}+\bar{v}+n)}+\frac{4\Gamma(\bar{v}-1)^2\Gamma(2\bar{u}+n+2)\Gamma(2\bar{u}+\bar{v}+n+1)}{\bar{u}\Gamma(2\bar{u}+1)\Gamma(\bar{v}+n-1)\Gamma(\bar{v}+n+2)\Gamma(2\bar{u}+\bar{v}+n)}+\\
&\frac{4\bar{v}\bar{u}\Gamma(\bar{v})^2\Gamma(2\bar{u}+n+1)\Gamma(2\bar{u}+\bar{v}+n+1)}{(2\bar{u}-1)\Gamma(2\bar{u}+1)\Gamma(\bar{v}+n+2)\Gamma(\bar{v}+n)\Gamma(2\bar{u}+\bar{v}+n)}+\frac{2\bar{v}\Gamma(\bar{v})^2\Gamma(2\bar{u}+n+2)}{\bar{u}\Gamma(2\bar{u}+1)\Gamma(\bar{v}+n+2)}\times\\
&\frac{\Gamma(2\bar{u}+\bar{v}+n+1)}{\Gamma(\bar{v}+n)\Gamma(2\bar{u}+\bar{v}+n+1)}+\frac{8\bar{u}\Gamma(\bar{v})^2\Gamma(2\bar{u}+n+1)\Gamma(2\bar{u}+\bar{v}+n+1)}{(2\bar{u}-1)\Gamma(2\bar{u}+1)\Gamma(\bar{v}+n+2)\Gamma(\bar{v}+n)\Gamma(2\bar{u}+\bar{v}+n)}
\end{aligned}\right]. \tag{45}$$

To obtain the Fisher information for momentum space, the same steps for Shannon entropy in the momentum space were strictly followed to have

$$I(\gamma)=N_{n,\ell}^2\left[I_{\gamma_1}+I_{\gamma_2}+I_{\gamma_3}+I_{\gamma_4}+I_{\gamma_5}+I_{\gamma_6}\right], \tag{46}$$

Where

$$\left.\begin{aligned}
&I_{\gamma_1}=\left(\frac{1-2\bar{u}}{2}\right)^2\int_{-1}^{1}\left(\frac{1-x}{2}\right)^{2\bar{u}-3}\left(\frac{1+x}{2}\right)^{3\bar{v}+3}\left[P_n^{(2\bar{u},\bar{v})}(x)\right]^2dx, I_{\gamma_2}=\left(\frac{1+v}{2}\right)^2\int_{-1}^{1}\left(\frac{1-x}{2}\right)^{2\bar{u}-1}\left(\frac{1+x}{2}\right)^{\bar{v}+1}\left[P_n^{(2\bar{u},\bar{v})}(x)\right]^2dx,\\
&I_{\gamma_3}=3.9204\int_{-1}^{1}\left(\frac{1-x}{2}\right)^{2\bar{u}-1}\left(\frac{1+x}{2}\right)^{\bar{v}+1}\left[P_n^{(2\bar{u},\bar{v})}(x)\right]^2dx, I_{\gamma_4}=\left(\frac{1-2\bar{u}}{2}\right)\left(\frac{1+\bar{v}}{2}\right)\int_{-1}^{1}\left(\frac{1-x}{2}\right)^{2\bar{u}-2}\left(\frac{1+x}{2}\right)^{\bar{v}}\left[P_n^{(2\bar{u},\bar{v})}(x)\right]^2dx,\\
&I_{\gamma_5}=-0.99(1-2\bar{u})\int_{-1}^{1}\left(\frac{1-x}{2}\right)^{2\bar{u}-2}\left(\frac{1+x}{2}\right)^{\bar{v}+1}\left[P_n^{(2\bar{u},\bar{v})}(x)\right]^2dx, I_{\gamma_6}=-0.99(1+\bar{v})\int_{-1}^{1}\left(\frac{1-x}{2}\right)^{2\bar{u}-1}\left(\frac{1+x}{2}\right)^{\bar{v}}\left[P_n^{(2\bar{u},\bar{v})}(x)\right]^2dx.
\end{aligned}\right\}. \tag{47}$$

Using integral in the Appendix B, the Fisher information for the momentum space is given as

$$\left.\begin{aligned}
&I_{\gamma_1}=\left(\frac{1-2\bar{u}}{2}\right)^2\int_{-1}^{1}\left(\frac{1-x}{2}\right)^{2\bar{u}-3}\left(\frac{1+x}{2}\right)^{3\bar{v}+3}\left[P_n^{(2\bar{u},\bar{v})}(x)\right]^2dx, I_{\gamma_2}=\left(\frac{1+v}{2}\right)^2\int_{-1}^{1}\left(\frac{1-x}{2}\right)^{2\bar{u}-1}\left(\frac{1+x}{2}\right)^{\bar{v}+1}\left[P_n^{(2\bar{u},\bar{v})}(x)\right]^2dx,\\
&I_{\gamma_3}=3.9204\int_{-1}^{1}\left(\frac{1-x}{2}\right)^{2\bar{u}-1}\left(\frac{1+x}{2}\right)^{\bar{v}+1}\left[P_n^{(2\bar{u},\bar{v})}(x)\right]^2dx, I_{\gamma_4}=\left(\frac{1-2\bar{u}}{2}\right)\left(\frac{1+\bar{v}}{2}\right)\int_{-1}^{1}\left(\frac{1-x}{2}\right)^{2\bar{u}-2}\left(\frac{1+x}{2}\right)^{\bar{v}}\left[P_n^{(2\bar{u},\bar{v})}(x)\right]^2dx,\\
&I_{\gamma_5}=-0.99(1-2\bar{u})\int_{-1}^{1}\left(\frac{1-x}{2}\right)^{2\bar{u}-2}\left(\frac{1+x}{2}\right)^{\bar{v}+1}\left[P_n^{(2\bar{u},\bar{v})}(x)\right]^2dx, I_{\gamma_6}=-0.99(1+\bar{v})\int_{-1}^{1}\left(\frac{1-x}{2}\right)^{2\bar{u}-1}\left(\frac{1+x}{2}\right)^{\bar{v}}\left[P_n^{(2\bar{u},\bar{v})}(x)\right]^2dx.
\end{aligned}\right\}. \tag{47}$$

$$I(\gamma)=4\pi\left[\begin{aligned}
&\frac{(1-2\bar{u})^2\Gamma(2\bar{u}+n-2)\Gamma(3\bar{v}+n+4)\Gamma(2\bar{u}+\bar{v}+n+1)}{2(4\bar{u}-4)\Gamma(2\bar{u}+1)\Gamma(\bar{v}+n+2)\Gamma(2\bar{u}+3\bar{v}+n+1)}+\frac{(1+\bar{v})^2\Gamma(2\bar{u}+n)}{4(2\bar{u}-1)\Gamma(2\bar{u}+1)}+\\
&\frac{3.9204\Gamma(2\bar{u}+n)}{(2\bar{u}-1)\Gamma(2\bar{u}+1)}+\frac{(1-2\bar{u})(1+\bar{v})\Gamma(2\bar{u}+n-1)\Gamma(\bar{v}+n+1)\Gamma(2\bar{u}+\bar{v}+n+1)}{4(2\bar{u}-2)\Gamma(2\bar{u}+1)\Gamma(\bar{v}+n+2)\Gamma(2\bar{u}+\bar{v}+n-1)}-\\
&\frac{0.99\Gamma(2\bar{u}+\bar{v}+n+1)}{\Gamma(2\bar{u}+1)\Gamma(2\bar{u}+\bar{v}+n)}\left[\frac{(1-2\bar{u})\Gamma(2\bar{u}+n-1)\Gamma}{(2\bar{u}-2)}+\frac{(1+\bar{v})\Gamma(2\bar{u}+n)\Gamma(\bar{v}+n+1)}{(2\bar{u}-1)\Gamma(\bar{v}+n+2)}\right]
\end{aligned}\right]. \tag{48}$$

6. Discussion

The variation of Shannon entropy for position space and momentum space against the parameter A are shown in Figures 1. A decrease in Shannon entropy for the momentum space corresponds to an increase in Shannon entropy for the position space and vice visa. As the parameter A increases, the Shannon entropy for position space featured out entropy squeezing. In the position space, there are more concentration of the wave function of the state as the parameter A increases resulting to a greater stability of the system. In Figure 2, the plots of Fisher information for position space and momentum space respectively against A are shown. As the parameter A increases in the position space, there is a decrease in the uncertainty of the system which increases the accuracy for predicting the localization of a particle in the system. This result is found to be opposite in the variation of the Fisher information for momentum space against the parameter A. The results for Shannon entropy and Fisher information respectively physically showed that a diffused density distribution γ(p) in momentum space is associated with a localized density distribution $\rho(r)$ in the position configuration.

The comparison of the energy eigenvalues with the previous results is shown in Table 1. The two results are in fair agreement. The fairness of the agreement gets weaker as the quantum number increases linearly. Table 2 shows the presentation of the numerical computations of Shannon entropy for both the position space and momentum space. The results justified and confirmed Beckner, Bialynicki-Birula and Mycieslki (BBM) $S(\rho)+S(\gamma)\geq D(1+log\pi)$. In this case, D = 1. Thus, the right hand side is $1+log\pi$ = 1.497206180. This simply means that the sum of the entropies cannot go beyond 1.497206180. The minimum bond for the sum of the entropies from the Table is 11.58371953, which is greater than 1.497206180. From Table 2, it can be seen that the position Shannon entropy exhibits entropy squeezing as the quantum number increases linearly. The results of the position space and momentum space Fisher information are presented in Table 3. The two Fisher information varies inversely with one another. A squeezing effect is noted in the position space as αincreases steadily. The results in the Table 3 also confirmed the popular Cramer Rao uncertainty inequality $I(\rho)I(\gamma)\geq 36.$ This inequality shows that the product of the Fisher information cannot go beyond 36. In our own case, as shown in Table 3, the minimum bound for the product of Fisher information is 49.88691549 which is greater than 36. This proves the accuracy of any result. The physical meaning of the inequality is that a decrease in Shannon entropy for position space corresponds to an increase in Shannon entropy for momentum space, this also applies to the Fisher information. Thus, a diffused density distribution γ(p) in momentum space is associated with a localized density distribution $\rho(r)$ in the position space or configuration space.

7. Conclusions

The study examined the manner of the energy eigenvalues, the wave function, Shannon entropy and Fisher information under the background of approximate solutions of

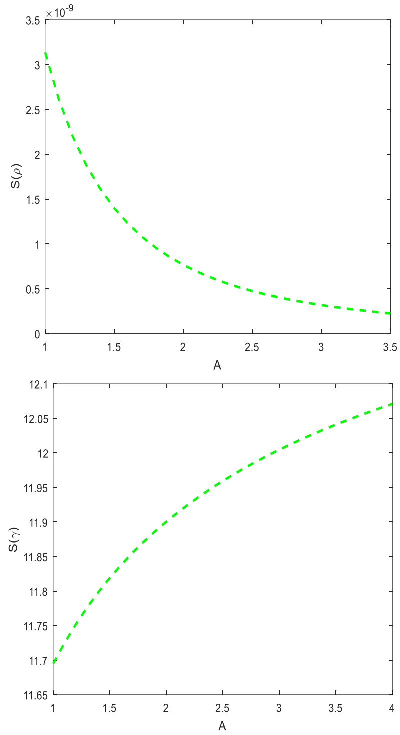

Figure 1. Shannon entropy for momentum space and position space respectively against A with $\ell=\mu=\hbar=B=v_0=1$ and $\delta=0.15$.

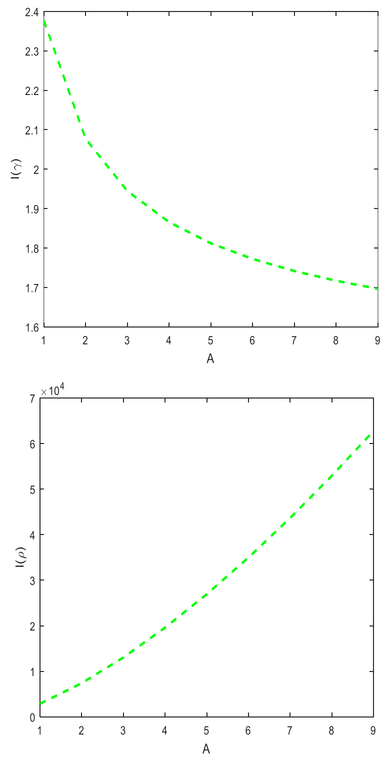

Figure 2. Fisher information for momentum space and position space respectively against A with $\ell=\mu=\hbar=B=v_0=1$, $\alpha=5$ and $\delta=0.15$.

Table 1. Comparison of the energy eigenvalues of the Trigonometric Inversely Quadratic plus Coulombic Hyperbolic Potential with $\mu=\hbar=\ell=v_0=A=1,\ B=0.2$ and $\delta=\alpha=0.1.$

n	ℓ	SUSY	(Okon *et al.*, 2020b)	*n*	ℓ	SUSY	(Okon *et al.*, 2020b)
0	0	0.202574496	0.1988204727	0	1	0.207595873	0.1999999965
1	0	0.202563337	0.1987136453	1	1	0.207602790	0.1965293766
2	0	0.200903711	0.1931126507	2	1	0.206358327	0.1887438717
3	0	0.198825946	0.1845598182	3	1	0.204612314	0.1779319895
4	0	0.196579945	0.1733752330	4	1	0.202615251	0.1644126026
5	0	0.194249554	0.1596385566	5	1	0.200474627	0.1482951488
6	0	0.191870838	0.1433772158	6	1	0.198244234	0.1296253900
7	0	0.189461875	0.1246027028	7	1	0.195953973	0.1084252044
8	0	0.187032725	0.1033205198	8	1	0.193621791	0.0847061215

Table 2. Shannon entropy for various quantum number with $\delta=0.15,\quad \ell=\mu=\hbar=1,\ \alpha=5$ and $B=A=v_0=1$ for the confirmation of Beekner, Bialynicki-Birula and Mycieslki (BBM) principle $S(\rho)+S(\gamma)\geq 1+log\pi.$

n	$S(\rho)$	$S(\gamma)$	$S(\rho)+S(\gamma)$
0	2.614671605	11.54216753	14.15683914
1	0.045045859	11.56295940	11.60800526
2	0.000791654	11.58292787	11.58371953
3	0.000014017	11.60212092	11.60213494
4	0.000000248	11.62058285	11.62058310
5	0.000000004	11.63835465	11.63835465

Table 3. Fisher information at the ground state for position space and momentum space with $\ell=\hbar=\mu=v_0=1,$ $n=0,\ \delta=0.05,\ A=-0.1$ and $B=1$ for the confirmation of Cramer Rao relation $I(\rho)I(\gamma)\geq 36.$

α	$I(\rho)$	$I(\gamma)$	$I(\rho)I(\gamma)$
1	1.561567081	31.94670027	49.88691549
2	0.109438724	477.0642202	52.20929952
3	0.066042933	893.9385694	59.03832505
4	0.028099201	2491.174037	69.99999999
5	0.008564905	9573.153243	81.99314808
6	0.004963984	21652.88253	107.4845624

a one-dimensional Schrödinger equation with Trigonometric Inversely Quadratic plus Coulombic Hyperbolic Potential. The results of the energy eigenvalues agreed with the results of Okon *et al.* (2020b). The detailed study of the Shannon entropy and Fisher information gave the results that obeyed the BBM principle and Cramer Rao uncertainty inequality. These results are found to be in excellent agfigureement with those of the literature, thus, the application of information entropic measures under the Trigonometric Inversely Quadratic plus Coulombic Hyperbolic Potential is possible.

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