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Original Article

Some models for inverse minimum spanning tree problem with uncertain edge weights

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Abstract

The inverse minimum spanning tree (IMST) problem is an inverse optimization problem in which one makes the least modification to the edge weights of a predetermined spanning tree, to make it the minimum spanning tree with respect to new edge weights. For a deterministic environment, the problem has been extensively studied. In an uncertain environment, the problem has been studied previously using stochastic edge weights or fuzzy edge weights. However, in the absence of enough data, approximation of a random variable is not possible. Further, the unobservable nature of edge weights means that assignment of fuzzy weights is also not possible. In this situation, the assignment of edge weights is done based on belief degree of some experts in the field. To deal with the problem of belief degree, the uncertainty theory is mostly suited. In this paper, two specific models for inverse minimum spanning tree are initiated, taking rough variables and uncertain normal variables as edge weights. Based on the properties of uncertainty, two specific models are formulated for the inverse minimum spanning tree problem. The models are converted to their equivalent deterministic models, which are solved by some standard optimization method. A numerical example is given to illustrate the model and its solution.

Keywords: minimum spanning tree, uncertain minimum spanning tree, rough minimum spanning tree, inverse optimization, uncertainty theory

1. Introduction

In the IMST problem, the objective is to make a modification to the weights of edges in a connected weighted graph, so that a predetermined spanning tree becomes an MST and the total modification of the weights should be minimal. The inverse minimum spanning tree (IMST) problem is an inverse optimization problem. This problem has great importance due to its immense applications in high-speed communication, computerized tomography, conjoint analysis, behavioral decision making, geographical science, performance evaluation, etc., especially in reconstruction problems (Liu, 2002, 2004, 2010). The IMST problem was initialized by Zhang, Liu, and Ma (1996) who proposed a combinatorial method to solve the problem. Since then, much work has been done on the IMST problem, considering the weights of the edges in deterministic as well as in uncertain environments. Guan and Zhang (2007) considered a class of inverse constrained bottleneck problems under the weighted l_∞ norm. Wang (2012) proposed two models for the partial inverse most unbalanced spanning tree problems under the weighted Hamming distance and the weighted l_1 norm. The research works of Ahuja and Orlin (2000), He, Zhang, and Yao (2005), Hochaum (2003) and Zhang, Liu, and Ma (1996) made the IMST problem a well-developed inverse optimization problem.

Also, many other works have contributed a lot to clarifying and solving the IMST problem and some of its derivatives. Several efficient algorithms have been designed to solve the IMST problem with as low computational

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complexity as possible. Sokkalingam, Ahuja, and Orlin (1999) developed a specific shortest path algorithm with run time $O(n^3)$ that can solve the IMST problem. Subsequently Ahuja and Orlin (2000) improved the previous $O(n^3)$ algorithm to a more efficient $O(n^2 \log n)$ time algorithm. He *et al*., (2005) presented some strongly polynomial algorithms for the weighted IMST problem under the Hamming distance, and so on. In many practical circumstances, the edge weights cannot be explicitly determined, and many parameters related to the problem may not be fully determined. In view of this non-determined nature of the parameters, some researchers used probability theory or fuzzy theory to deal with the problem. Zhang and Zhou (2006) considered the IMST problem where the edge weights were taken as random variables and stochastic programming models with a hybrid intelligent algorithm were presented. Zhang *et al.,* (2014) studied the IMST problem with fuzzy edge weights. Dey, Pal, and Pal, (2016), Dey, Pradhan, Pal, and Pal (2018), Dey, Son Pal, and Long, (2020) have used the interval type-2 fuzzy set to represent the arc lengths and a new genetic algorithm was proposed to solve the fuzzy shortest path problem and fuzzy minimum spanning tree problem. Further, an MST of an undirected type-2 fuzzy weighted connected graph was investigated, where the edge weights were discrete type-2 fuzzy variables, by Dan, Majumder, Kar, and Kar (2021) and a modified type-2 fuzzy Boruvka's algorithm was proposed to determine the MST.

However, when no samples are available to estimate a probability distribution in the non-deterministic environment, the views of some domain experts are used to evaluate the belief degree that each event will occur. In order to deal with the belief degree rationally, the uncertainty theory was developed by Liu (2004) who subsequently applied the theory to model many problems under uncertain environment. Zhang *et al*., 2013 have proposed two uncertain programming models for the IMST problem, considering the edge weights as uncertain linear variables. A bi-objective rough-fuzzy quadratic MST problem has been studied for a connected graph where the linear and the quadratic edge weights are represented as rough variables, and a model was proposed by Majumdar, Kar, & Pal (2019) using rough-fuzzy chance constrained programming technique. Further, to deal with this type of uncertainty, Majumdar *et al*., (2020) have given two models, one an expected value model and the other one is a chance constrained model of uncertain multi-objective SPP for a weighted connected directed graph, and they formulated two multi-objective genetic algorithms to find the SPP. A Multi-Objective MST has been studied by Majumdar *et al*., (2022) with indeterminate edge weights. Two models of uncertain Multi-Objective MSTs were developed and their corresponding crisp equivalence models were investigated and solved using the epsilon-constraint method. Other approaches to model the uncertainty problems using fuzzy graphs, rough-fuzzy graph, m-polar fuzzy graphs and their applications are available in the literature (Akram, 2019; Akram, Sarwar & Dudek, 2021). Some more related work can be found also (Chakraborty, Mondal, Alam & Dey, 2021; Lakhwani, Mohanta, Dey, Mondal & Pal, 2022; Mohanta, Dey, Pal, Long & Son, 2020; Mondal, Dey, De & Pal, 2021; Xiao, Dey & Son, 2020).

In this paper, a specific IMST problem is analysed taking the edge weights as rough variables. A generalization of the path optimality condition provided by Ahuja and Orlin (2000) is developed to model the uncertain α-minimum spanning tree and subsequently the uncertain IMST. Two models are developed reducing the rough variable case to a deterministic one. In the first case rough variables are converted to deterministic values using uncertain normal distribution and the model is formed satisfying uncertain normal distribution properties, which gives a linear programming problem that is solved. In the second case using trust distribution, rough variables are converted to certain values for different confidence levels and another model for IMST is established for different confidence levels satisfying trust theory, which is a linear programming problem soluble using general methods of LPP.

The rest of the paper is organized as follows: In Section 2, classical deterministic inverse minimum spanning tree problems are discussed and some basic concepts of uncertainty theory, uncertain variables, and rough variables are presented. In Section 3, uncertain inverse minimum spanning tree problem is formulated. Two models of IMST are formed taking uncertain normal variables and another particular type of with rough variables. In Section 4, the equivalent crisp model for uncertain IMST is discussed and one numerical example is presented for illustration. In Section 5, the equivalent crisp model for rough IMST is discussed and solved in one numerical example. Finally, the conclusion is given in Section 6.

2. Preliminaries

In this section, the classical inverse shortest path problem is reviewed and some notions and results of uncertain variables and rough variables are given, which shall be used to handle the uncertain inverse shortest path.

2.1 Classical inverse minimum spanning tree problem

Let $G = (V, E)$ be a connected graph with vertex set $V = \{ v_1, v_2, v_3, \ldots, v_n\}$ and edge set $E = \{1, 2, \ldots, m\}$. Let c_i be the weight of the edge $i \in E$.

A spanning tree T^0 is said to be minimum spanning tree if $\sum_{i \in T^0} c_i \le \sum_{j \in T} c_j$ holds for any tree T. The edges of the given spanning tree T^0 are called the tree edges and the edges not in T^0 are called the non-tree edges. Hence, the set of all non-tree edges is $E \setminus T^0$. For any non-tree edge j, there exists a unique path in T^0 which is known as tree path of non-tree edge j and denoted by P_j.

The classic inverse minimum spanning tree (IMST) problem is to find some new edge weights such that the pre-determined spanning tree T^0 is a minimum spanning tree with respect to the new edge weights and the total modification of edge weights is at its minimum.

Let a new weight vector $x = (x_1, x_2, \ldots, x_i \ldots)$ be assigned to the edges. T^0 is a minimum spanning tree with respect to x and $\sum_{i=1}^{m} |x_i - c_i|$ is minimum.

Path optimality condition:
As proposed by Ahuja *et al*. [2], for a given connected graph G = (V,E) with edge weights x_i, $i \in E = \{1, 2,.., m\}$, a spanning tree T^0 is a minimum spanning tree if and only if,

$$x_i - x_j \leq 0 \quad \text{for } j \in E \setminus T^0,\ i \in P_j \tag{1}$$

where E/T^0 is the set of non-tree edges, and P_j is the tree path of edge j.
Using the above path optimality condition, the classical IMST problem can be formulated as

$$\begin{cases} \min \quad \sum_{i=1}^{m} |x_i - c_i| \\ \text{subject to} \quad x_i - x_j \leq 0, \quad j \in E \setminus T^0,\ i \in P_j \end{cases} \tag{2}$$

where c_i and x_i are the original and the new edge weights of each edge $i \in E$, respectively.

2.2 Uncertainty theory

B. Liu (Liu, 2004; Liu, 2010) has developed uncertainty theory which is considered a new approach to deal with indeterminacy factors when there is a lack of observed data. In this section, some basic concepts of uncertainty theory are reviewed that shall be used in this paper.

2.2.1 Uncertainty measure

Let L be a σ-algebra on a nonempty set Γ. A set function M: L $\longrightarrow$[0,1] is called an uncertain measure if it satisfies the following axioms
Axiom 1: (Normality axiom) M (Γ) = 1 for the universal set Γ
Axiom 2: (Duality axiom) $\mathrm{M}(\Lambda) + \mathrm{M}(\Lambda^c) = 1$ for every event Λ
Axiom 3: (Sub additivity axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, ...$ we have

$$\mathrm{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathrm{M}(\Lambda_i)$$

The triplet (Γ, L, M) is called an uncertainty space.
Axiom 4: (Product measure) Let $(\Gamma_k, L_k, \mathrm{M}_k)$ be uncertainty spaces for k = 1, 2……The product uncertain measure is an uncertain measure satisfying

$$\mathrm{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigcap_{k=1}^{\infty} \mathrm{M}(\Lambda_k)$$

where, Λ_k are arbitrary chosen events for L_k for k = 1,2…… respectively.

2.2.2 Uncertain variable

An uncertain variable ξ is an essentially a measurable function from an uncertainty space to the set of real numbers. Let ξ be an uncertain variable. Then the uncertainty distribution of ξ is defined as $\phi(x) = M\{\xi \leq x\}$ for any real number x.
Normal uncertain distribution:
An uncertain variable ξ is called normal if it has a normal uncertainty distribution

$$\phi(x) = \left(1 + \exp\left(\frac{\pi(e - x)}{\sqrt{3}\sigma}\right)\right)^{-1} \qquad for \ \ x \in R \tag{3}$$

Normal uncertainty distribution is denoted by N (e, σ), where e and σ are real numbers with σ > 0.
An uncertain distribution ϕ is said to be regular if its inverse function $\phi^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0,1)$.
The normal uncertainty distribution N (e, σ) is also regular and its inverse uncertainty distribution is

$$\phi(x) = e + \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} \tag{4}$$

2.3. Rough variable

The concept of a rough variable was introduced by Liu (Liu, 2004) as uncertain variable. The following definitions are based on Liu (Liu, 2002).

Definition 1. Let Λ be a non-empty set, A be σ- algebra of subsets of Λ, Δ be an element in A, and π be a non-negative, real-valued, additive set function on A. The quadruple (Λ, Δ, *A*, π) is called a rough space.

Definition 2. A rough variable ξ on the rough space Λ, Δ, *A*, π) is a measurable function from Λ to the set of real numbers $\mathfrak{R}$ such that for every Borel set B of $\mathfrak{R}$, we have $\{\lambda \in \Lambda \mid \ \xi(\lambda) \in \ \mathrm{B}\} \in A$.

Then the lower and upper approximation of the rough variable ξ are defined as follows

$$\bar{\zeta} = \{\zeta(\lambda) \big| \lambda \in \Lambda\} \qquad \text{(Upper approximation)}$$

$$\underline{\zeta} = \{\zeta(\lambda) \big| \lambda \in \Delta\} \qquad \text{(Lower approximation)}$$

Definition 3. ([a, b], [c, d]) with c ≤ a < b ≤ d is a rough variable, where ξ (λ) = λ from the rough space to the set of real numbers and $\Lambda = \{\lambda \big| \ c \le \lambda \le d\} \ \ and \ \ \Delta = \{\lambda \ \big| \ a \le \lambda \le b\}$, A is the Borel algebra on Λ, and π is the Lebesgue measure.

Definition 4: Let (Λ, Δ, *A*, π) be a rough space. Then the upper and lower trust of event A are defined by

$$\mathrm{T}\underline{\mathrm{r}}(\mathrm{A}) = \frac{\pi\{\mathrm{A}\}}{\pi\{\Lambda\}} \quad \text{and} \quad \mathrm{T}\overline{\mathrm{r}}(\mathrm{A}) = \frac{\pi\{\mathrm{A} \cap \Delta\}}{\pi\{\Lambda\}}$$

The trust of the event A is defined as

$$\mathrm{Tr}(\mathrm{A}) = \frac{1}{2}\left(\mathrm{T}\underline{\mathrm{r}}(\mathrm{A}) + \ \mathrm{T}\,\overline{\mathrm{r}}(\mathrm{A})\right)$$

Definition 5. The trust distribution function ϕ:(-∞, ∞) → [0,1] of a rough variable ξ is defined as $\phi(x) = Tr\{\lambda \in \Lambda \ \big| \ \zeta(\lambda) \le x\}$. That is, ϕ(x) is the trust that the rough variable ζ takes a value less than or equal to x.

Definition 6. For a given value of r and ξ = ([a, b], [c, d]), the trust of rough events characterized by $\zeta \le$ r and $\zeta \ge$ r is given by the following expressions respectively

$$Tr\{\zeta \le r\} = \begin{cases} 0 & if\ r \le c \\ \dfrac{r-c}{2(d-c)} & if\ c \le r \le a \\ \dfrac{1}{2}\left(\dfrac{r-a}{b-a} + \dfrac{r-c}{d-c}\right) & if\ a \le r \le b \\ \dfrac{1}{2}\left(\dfrac{r-c}{d-c} + 1\right) & if\ b \le r \le d \\ 1 & if\ r \ge d \end{cases} \tag{5}$$

$$Tr\{\zeta \ge r\} = \begin{cases} 0 & if\ r \ge d \\ \dfrac{d-r}{2(d-c)} & if\ b \le r \le d \\ \dfrac{1}{2}\left(\dfrac{d-r}{d-c} + \dfrac{b-r}{b-a}\right) & if\ a \le r \le b \\ \dfrac{1}{2}\left(\dfrac{d-r}{d-c} + 1\right) & if\ c \le r \le a \\ 1 & if\ r \le c \end{cases} \tag{6}$$

The trust measure satisfies the following:

Definition 7. Let ξ be rough variables defined on the rough space (Λ, Δ, *A*, π). The expected value of ξ is defined by

$$\mathrm{E}(\xi) = \int_0^{\infty} \mathrm{Tr}\{\xi \ge r\}\, dr \ - \int_{-\infty}^{0} \mathrm{Tr}\{\xi \le r\}\, dr$$

Definition 8. The trust density function $f: \mathrm{R} \rightarrow [0,\infty)$ of a rough variable ξ is a function such that $\phi(x) = \int_{-\infty}^{x} f(y)dy$ holds for all x ∈ (-∞, ∞), where ϕ is trust distribution of ξ. If ξ = ([a, b], [c, d]) be a rough variable such that c ≤ a < b ≤ d, then the trust distribution ϕ(x) = Tr{ξ ≤ x} is

$$\phi(x) = \begin{cases} 0 & \text{if } x \le c \\ \dfrac{x-c}{2(d-c)} & \text{if } c \le x \le a \\ \dfrac{[(b-a)+(d-c)]x + 2ac - ad - bc}{2(b-a)(d-c)} & \text{if } a \le x \le b \\ \dfrac{x+d-2c}{2(d-c)} & \text{if } b \le x \le d \\ 1 & \text{if } x \ge d \end{cases} \tag{7}$$

and the trust density function is defined as

$$f(x) = \begin{cases} \dfrac{1}{2(d-c)} & \text{if } c \le x \le a \text{ or } b \le x \le d \\ \dfrac{1}{2(b-c)} + \dfrac{1}{2(d-c)} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

Definition 9. (**Expected value of a rough variable**)
Let the trust distribution φ of a rough variable ξ and trust density function f exist. Then the expected value or mean of ξ is defined as $E[\zeta] = \int_{-\infty}^{\infty} x\, f(x)\, dx$, provided the integral exists.

Definition 10. (**Variance of a rough variable**)
If ξ is a rough variable with finite expected value E[ξ], then the variance of ξ is defined as V[ξ] = E [(ξ – E[ξ])^2]

3. Problem Description and Model Formulation

3.1 Uncertain inverse minimum spanning tree

In this section, a specific IMST problem is investigated with uncertain edge weights. Initially, rough variables are assigned as uncertain edge weights to each i ∈ E and then these rough variables are approximated by uncertain normal variables for further investigation.

In order to provide a mathematical description of the problem, the following notions are used.
G = (V, E) be a connected graph with vertex set V = {v_1, v_2…, v_n} and edge set E = {1, 2,…, m}
T^0 is a pre-determined spanning tree of G which needs to be minimum spanning tree after modification of edge weights.
c_i are the original weight of the edge i ∈ E, i = 1, 2, … , m.
x_i is the decision variable representing the new edge weights of the edge i ∈ E.
ξ_i (x_i) is the uncertain edge weights with respect to x_i, i ∈ E.

As ξ_i are uncertain variables, the condition (1) for uncertain minimum spanning tree becomes invalid. So, for modelling uncertain IMST problem with respect to uncertain edge weights, the concept of α-minimum spanning tree as proposed by Zhang *et al*. is used.

Definition 11. (**Uncertain α-minimum spanning tree**)
Given a connected graph G = (V , E) with vertex set V = { v_1, v_2, … , v_n} and edge set E = {1, 2, …. , m} with uncertain edge weights ξ_i, i ∈ E and a given confidence level α, a spanning tree T^0 is said to be an uncertain α-minimum spanning tree if

$$\mathrm{M}\left\{ \sum_{i \in T^0} \zeta_i \le \sum_{j \in T} \zeta_j \right\} \ge \alpha \tag{9}$$

holds for any spanning tree T.

3.2 Uncertain path optimality condition

For any connected graph G = (V,E) with uncertain edge weights ξ_i, $i \in E$ and a confidence level α , a spanning tree T^0 is an uncertain α-minimum spanning tree with respect to uncertain edge weights if and only if

$$M\{\xi_i(x) \le \xi_j(x)\} \ge \alpha, \quad j \in E \setminus T^0,\ i \in P_j \tag{10}$$

where, $E \setminus T^0$ is the set of non-tree edges and P_j is the tree path of edge j.
So the uncertain IMST problem can be formulated as follows

$$\begin{cases} \min \quad \sum_{i=1}^{m} |x_i - c_i| \\ subject\ to \\ M\{\zeta_i(x_i) \le \zeta_j(x_j)\} \ge \alpha, \quad j \in E \setminus T^0,\ i \in P_j \\ x_i \ge 0, \qquad i = 1,2,...,m \end{cases} \tag{11}$$

where, α is a pre-determined confidence level.

3.3 Crisp equivalent model

In order to solve the model (9), it is required to convert the model into its equivalent crisp model.
Let ξ be an uncertain variable. Then the uncertainty distribution of ξ is defined as $\phi(x) = M\{\xi \le x\}$ for any real number x. The model (9) can be converted to crisp equivalent form as

$$\begin{cases} \min \quad \sum_{i=1}^{m} |x_i - c_i| \\ subject\ to \\ \phi_i^{-1}(x_i,\alpha) \le \phi_j^{-1}(x_j,\alpha), \quad j \in E \setminus T^0,\ i \in P_j \\ x_i \ge 0, \qquad i = 1,2,...,m \end{cases} \tag{12}$$

where ϕ_i^{-1} represents the inverse distribution of uncertain variable ξ_i.
In this work, initially the edge weights are uncertain variables as rough variables of the form

$$([x_i - d_i, x_i + d_i], [x_i - 2d_i, x_i + 2d_i]). \tag{13}$$

In order to find the mean and variance of the rough variable, the following theorem is proposed.

Theorem 1. If ξ = ([a, b], [c, d]) is an uncertain variable with c ≤ a < b ≤ d, the mean and standard deviation of ξ are respectively (a + b + c +d)/4 and ($a^2 + b^2 + c^2 + d^2$ + ab + cd – $6e^2$)/6, where e is the mean.

Proof. Using the trust density function f(x) as per definition

$$E(\xi) = \int_{-\infty}^{\infty} x f(x) dx$$
$$= \int_c^a \frac{x}{2(d-c)} dx + \int_a^b [\frac{1}{2(b-a)} + \frac{1}{2(d-c)}] x dx + \int_b^d \frac{x}{2(d-c)} dx$$
$$= (a+b+c+d)/4 = e$$
$$Var(\xi) = E(\xi - E(\xi))^2$$
$$= \int_c^a \frac{(x-e)^2}{2(d-c)} dx + \int_a^b [\frac{1}{2(b-a)} + \frac{1}{2(d-c)}](x-e)^2 dx + \int_b^d \frac{(x-c)^2}{2(d-c)} dx$$
$$= (a^2 + b^2 + c^2 + d^2 + ab + dc - 6e^2)/6$$

From the above theorem, the mean and SD of the rough variable of the form (7) can be calculated as x_i and √5 d_i.

An empirical study revealed that the problem under uncertain edge weights, the uncertainty can be best studied using normal probability distribution. But, in the absence of a sufficient sample of observations, where the subjective estimation of experts is required, it is more likely that the experts give their valuable estimation in the form of range of values, which can be characterized by rough variables. For further analysis, the subjective estimation in the form of rough variables can be approximated by uncertain normal variables with mean and SD calculated using theorem 1.

Hence, from (4), $\phi_i^{-1}(x_i,\alpha) = x_i + \frac{\sqrt{15} d_i}{\pi} \ln\left(\frac{\alpha}{1-\alpha}\right)$.

So, the model (12) is reduced to the form

$$\begin{cases} \min \quad \sum_{i=1}^{m} |x_i - c_i| \\ \textit{subject to} \\ x_i + \frac{\sqrt{15} d_i}{\pi} \ln\left(\frac{\alpha}{1-\alpha}\right) \le x_j + \frac{\sqrt{15} d_j}{\pi} \ln\left(\frac{\alpha}{1-\alpha}\right), \quad j \in E \setminus T^0,\ i \in P_j \\ x_i \ge 0, \quad i = 1, 2, \ldots, m \end{cases} \tag{14}$$

Further, by introducing two auxiliary variables x_i^+ and x_i^- , the model can further be simplified, where

$$x_i^+ = \begin{cases} x_i - c_i & if \quad x_i \ge c_i \\ 0 & if \quad x_i < c_i \end{cases} \text{ and}$$

$$x_i^- = \begin{cases} 0 & if \quad x_i \ge c_i \\ c_i - x_i & if \quad x_i < c_i \end{cases}.$$

So, $|x_i - c_i| = x_i^+ + x_i^-$

$x_i = x_i^+ - x_i^- + c_i$

$x_i^+ \geq 0$

$0 \leq x_i^- \leq c_i$.

So the model (12) can be transformed to the form

$$\begin{cases} \min \quad \sum_{i=1}^{m} x_i^+ + \sum_{i=1}^{m} x_i^- \\ \textit{subject to} \\ x_i^+ - x_i^- + c_i + \frac{\sqrt{15} d_i}{\pi} \ln\left(\frac{\alpha}{1-\alpha}\right) \le x_j^+ - x_j^- + c_j + \frac{\sqrt{15} d_j}{\pi} \ln\left(\frac{\alpha}{1-\alpha}\right), \quad j \in E \setminus T^0,\ i \in P_j \\ x_i \ge 0, \quad i = 1, 2, \ldots, m \\ 0 \le x_i^- \le c_i \end{cases} \tag{15}$$

The model (15) is a deterministic linear programming problem and can be solved by any standard technique

3.4 Numerical example

In this section, one numerical example is taken to implement the proposed model. In the following figure, one traffic network is considered having six vertices as traffic hub and ten edges as roads. The spanning tree (b, a), (b, d), (b, c), (c, e), (c, f) is taken as the predetermined spanning tree which is to be converted to minimum spanning tree after modification of the weights. There are three weights on each road, where c_i and x_i are the original and new width of the road i, and ξ_i denotes the uncertain travelling time on road i, which are assumed to be rough variables with respect to x_i. The level of confidence α is taken as 0.9. In Table 1, detailed data are shown.

Table 1. Data table

Edge	Edge no	Original parameter (c_i)	Uncertain edge weight($\xi_i(x_i)$)	mean	SD	$\Phi^{-1}(x_i, \alpha)$
(a, b)	1	100	$([x_1-10, x_1+10], [x_1-20, x_1+20])$	x_1	$10\sqrt{5}$	$x_1 + 27.1$
(b, d)	2	50	$([x_2-10, x_2+10], [x_2-20, x_2+20])$	x_2	$10\sqrt{5}$	$x_2 + 27.1$
(b, c)	3	60	$([x_3-10, x_3+10], [x_3-20, x_3+20])$	x_3	$10\sqrt{5}$	$x_3 + 27.1$
(c, e)	4	130	$([x_4-10, x_4+10], [x_4-20, x_4+20])$	x_4	$10\sqrt{5}$	$x_4 + 27.1$
(c, f)	5	140	$([x_5-10, x_5+10], [x_5-20, x_5+20])$	x_5	$10\sqrt{5}$	$x_5 + 27.1$
(a, d)	6	60	$([x_6-10, x_6+10], [x_6-20, x_6+20])$	x_6	$10\sqrt{5}$	$x_6 + 27.1$
(d, e)	7	80	$([x_7-10, x_7+10], [x_7-20, x_7+20])$	x_7	$10\sqrt{5}$	$x_7 + 27.1$
(d, c)	8	50	$([x_8-10, x_8+10], [x_8-20, x_8+20])$	x_8	$10\sqrt{5}$	$x_8 + 27.1$
(b, e)	9	160	$([x_9-10, x_9+10], [x_9-20, x_9+20])$	x_9	$10\sqrt{5}$	$x_9 + 27.1$
(e, f)	10	120	$([x_{10}-10, x_{10}+10], [x_{10}-20, x_{10}+20])$	x_{10}	$10\sqrt{5}$	$x_{10} + 27.1$

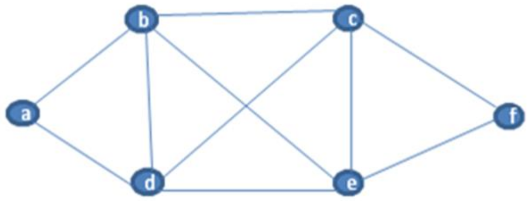

Figure 1. Road network

From the above data, the following uncertain programming model is obtained

$$\begin{cases} \min \quad \sum_{i=1}^{10} x_i^+ + \sum_{i=1}^{10} x_i^- \\ \textit{subject to} \\ x_i^+ - x_i^- + c_i + \frac{\sqrt{15} d_i}{\pi} \ln 9 \le x_j^+ - x_j^- + c_j + \frac{\sqrt{15} d_j}{\pi} \ln 9 \ , \quad j \in E \backslash T^0, \ i \in P_j \\ x_i^+ \ge 0, \qquad i = 1, 2, \ldots, m \\ 0 \le x_i^- \le c_i \end{cases} \tag{16}$$

where the non-tree edge set E \ T^0 = {6, 7, 8, 9, 10}, P_j is the tree path of non-tree edge j.
As d_i is taken uniformly as 10 for each i =1, 2..., 10, the model is transformed to the form

$$\begin{cases} \min \quad \sum_{i=1}^{10} x_i^+ + \sum_{i=1}^{10} x_i^- \\ \textit{subject to} \\ x_i^+ - x_i^- + c_i \le x_j^+ - x_j^- + c_j \ , \quad j \in E \backslash T^0, \ i \in P_j \\ x_i^+ \ge 0, \qquad i = 1, 2, \ldots, m \\ 0 \le x_i^- \le c_i \end{cases} \tag{17}$$

which which is a linear programming model. The solution to the model is

x_i^+ = {0, 0, 0, 0, 0, 0, 14, 74, 24, 0, 34}

x_i^- = {50, 0, 10, 0, 10, 0, 0, 0, 0, 0 }

x_i = {50, 50, 50, 130, 130, 74, 154, 74, 160, 154}.

The total modification is 216.

4. Rough Inverse Minimum Spanning Tree

In Section 3.1, if ξ_i are taken as rough variables, the condition (1) for uncertain minimum spanning tree becomes invalid. So, for modelling uncertain IMST problem with respect to uncertain edge weights, the concept of rough α-minimum spanning tree is proposed

Definition 12. (**Rough α-minimum spanning tree**)
Given a connected graph G = (V , E) with vertex set V = { $v_1, v_2, \ldots, v_n$} and edge set E = {1, 2, …. , m} with rough edge weights ξ_i, i ∈ E and a given confidence level α, a spanning tree T^0 is said to be an rough α-minimum spanning tree if

$$\text{Tr}\left\{ \sum_{i \in T^0} \zeta_i \le \sum_{j \in T} \zeta_j \right\} \ge \alpha \tag{18}$$

holds for any spanning tree T.

4.1 Rough path optimality condition

For any connected graph G = (V,E) with rough edge weights ξ_i, i ∈ E and a confidence level α, a spanning tree T^0 is a rough α-minimum spanning tree with respect to rough edge weights if and only if

$Tr\{\xi_i(x) \le \xi_j(x)\} \ge \alpha, \quad j \in E \setminus T^0, \; i \in P_j$

where, $E \setminus T^0$ is the set of non-tree edges and P_j is the tree path of edge j.
So the rough IMST problem can be formulated as follows

$$\begin{cases} \min \quad \sum_{i=1}^{m} |x_i - c_i| \\ subject\ to \quad Tr\{\zeta_i(x_i) \le \zeta_j(x_j)\} \ge \alpha, \quad j \in E \setminus T^0, \; i \in P_j \\ x_i \ge 0, \qquad i = 1, 2, ..., m \end{cases} \tag{19}$$

where, α is a pre-determined confidence level.

4.2. Crisp equivalent model

In order to solve the model (19), it is required to convert the model into its equivalent crisp model. In this model the rough weight of the edge i is taken as $\xi_i = ([x_i - d_i, x_i + d_i], [x_i - 2d_i, x_i + 2d_i]$, where d_i is a positive constant.

Hence, $\xi_i - \xi_j = ([x_i - x_j - d_i - d_j, x_i - x_j + d_i + d_j], [x_i - x_j - 2d_i - 2d_j, x_i - x_j + 2d_i + 2d_j])$
$= ([a, b], [c, d])$ (say)

Theorem 2. If $\xi = ([a, b], [c, d])$ is a rough variable with $c \le a < b \le d$, then for a predetermined confidence level $\alpha, (0 < \alpha \le 1)$, $\{Tr\{\xi \le r\} \ge \alpha\}$ is equivalent to

$$(i)\ (1-2\alpha)c + 2\alpha d \le r \qquad if\ \alpha \le \frac{a-c}{2(d-c)}$$

$$(ii)\ 2(1-\alpha)c + (2\alpha - 1)d \le r \qquad if\ \alpha \ge \frac{b+d-2c}{2(d-c)}$$

$$(iii)\ \frac{c(b-a) + a(d-c) + 2\alpha(b-a)(d-c)}{(b-a)+(d-c)} \le r \qquad otherwise$$

After simplification, (6) is equivalent to the following form

(i) $x_i - x_j \le 2(1 - 4\alpha)(d_i + d_j)$ if $\alpha \le 1/8$
(ii) $x_i - x_j \le 2(3 - 4\alpha)(d_i + d_j)$ if $\alpha \ge 7/8$
(iii) $x_i - x_j \le (4/3 - 8\alpha/3)(d_i + d_j)$ if $1/8 < \alpha < 7/8$

Hence, depending on the values of α, three cases shall arise.
Case 1: When $\alpha \le 1/8$
The model (19) can be converted to crisp equivalent form as

$$\begin{cases} \min \quad \sum_{i=1}^{m} |x_i - c_i| \\ subject\ to \quad x_i - x_j \le 2(1-4\alpha)(d_i + d_j), \quad j \in E \setminus T^0, \; i \in P_j \\ x_i \ge 0, \qquad i = 1, 2, ..., m \end{cases} \tag{*}$$

Case 2: When $\alpha \ge 7/8$, the model (7) can be converted to crisp equivalent form as

$$\begin{cases} \min \quad \sum_{i=1}^{m} |x_i - c_i| \\ subject\ to \quad x_i - x_j \le 2(1-4\alpha)(d_i + d_j), \quad j \in E \setminus T^0, \; i \in P_j \\ x_i \ge 0, \qquad i = 1, 2, ..., m \end{cases} \tag{**}$$

Case 3: When $1/8 < \alpha < 7/8$, the model (7) can be converted to crisp equivalent form as

$$\begin{cases} \min \quad \sum_{i=1}^{m} |x_i - c_i| \\ subject\ to \quad x_i - x_j \le \frac{4}{3}(1-2\alpha)(d_i + d_j), \quad j \in E \setminus T^0, \; i \in P_j \\ x_i \ge 0, \qquad i = 1, 2, ..., m \end{cases} \tag{***}$$

Further, by introducing two auxiliary variables x_i^+ and x_i^- , the model can further be simplified, where

$$x_i^+ = \begin{cases} x_i - c_i & if \quad x_i \ge c_i \\ 0 & if \quad x_i < c_i \end{cases} \text{ and } x_i^- = \begin{cases} 0 & if \quad x_i \ge c_i \\ c_i - x_i & if \quad x_i < c_i \end{cases}.$$

So, $\left| x_i - c_i \right| = x_i^+ + x_i^-$

$$x_i = x_i^+ - x_i^- + c_i$$

$x_i^+ \ge 0$

$0 \le x_i^- \le c_i$.

Hence, the model (*) can be transformed to

$$\begin{cases} \min \quad \sum_{i=1}^{m} x_i^+ + \sum_{i=1}^{m} x_i^- \\ subject \quad to \quad x_i^+ - x_i^- - x_j^+ + x_j^- \le 2 \ (1-4\alpha) \ (d_i + d_j) - c_i - c_j \ , \quad j \in E \ \setminus \ T^0, \quad i \in P_j \\ x_i \ge 0, \qquad i = 1, \ 2, \ \ldots \ , \ m \\ 0 \le x_i^- \le c_i \qquad i = 1, 2, \ \ldots, \ m \end{cases} \tag{20}$$

So the model (**) can be transformed to

$$\begin{cases} \min \quad \sum_{i=1}^{m} x_i^+ + \sum_{i=1}^{m} x_i^- \\ subject \quad to \quad x_i^+ - x_i^- - x_j^+ + x_j^- \le 2 \ (3-4\alpha) \ (d_i + d_j) - c_i - c_j \ , \quad j \in E \ \setminus \ T^0, \quad i \in P_j \\ x_i \ge 0, \qquad i = 1, \ 2, \ \ldots \ , \ m \\ 0 \le x_i^- \le c_i \qquad i = 1, 2, \ \ldots, \ m \end{cases} \tag{21}$$

So the model (***) can be transformed to

$$\begin{cases} \min \quad \sum_{i=1}^{m} x_i^+ + \sum_{i=1}^{m} x_i^- \\ subject \quad to \quad x_i^+ - x_i^- - x_j^+ + x_j^- \le \frac{4}{3} \ (1-2\alpha) \ (d_i + d_j) - c_i - c_j \ , \quad j \in E \ \setminus \ T^0, \quad i \in P_j \\ x_i \ge 0, \qquad i = 1, \ 2, \ \ldots \ , \ m \\ 0 \le x_i^- \le c_i \qquad i = 1, 2, \ \ldots, \ m \end{cases} \tag{22}$$

The models (20), (21) and (22) are deterministic linear programming problems and can solved by using any standard technique.

4.3 Numerical example

In this section, one numerical example is taken to implement the proposed model. In the following figure (Figure 2), one traffic network is considered having six vertices as traffic hub and ten edges as roads. The spanning tree ba, bd, bc, ce, cf is taken as the predetermined spanning tree which is to be converted to minimum spanning tree after modification of the weights. There are three weights on each road, where c_i and x_i are the original and new width of the road i, and ξ_i denotes the uncertain travelling time on road, which are assumed to be rough variables with respect to x_i. The level of confidence α is taken as 0.9.

Table 2. Edge weights

Edge	Edge No	Original parameter (c_i)	Uncertain edge weight ($\xi_i(x_i)$)
(a, b)	1	100	([x_1-10 , x_1+10] , [x_1-20 , x_1+20])
(b, d)	2	50	([x_2-10 , x_2+10] , [x_2-20 , x_2+20])
(b, c)	3	60	([x_3-10 , x_3+10] , [x_3-20 , x_3+20])
(c, e)	4	130	([x_4-10 , x_4+10] , [x_4-20 , x_4+20])
(c, f)	5	140	([x_5-10 , x_5+10] , [x_5-20 , x_5+20])
(a, d)	6	60	([x_6-10 , x_6+10] , [x_6-20 , x_6+20])
(d, e)	7	80	([x_7-10 , x_7+10] , [x_7-20 , x_7+20])
(d, c)	8	50	([x_8-10 , x_8+10] , [x_8-20 , x_8+20])
(b, e)	9	160	([x_9-10 , x_9+10] , [x_9-20 , x_9+20])
(e, f)	10	120	([x_{10}-10 , x_{10}+10] , [x_{10}-20 , x_{10}+20])

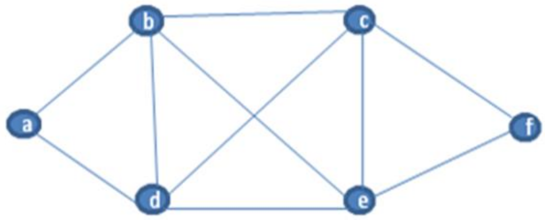

Figure 2. Traffic network

From the above data, the following programming model is obtained, where the non-tree edge set $E\backslash T^0 = \{6, 7, 8, 9, 10\}$, P_j is the tree path of non-tree edge j. As d_i is taken uniformly as 10 for each i = 1, 2,., 10, the model is transformed to the following form

$$\begin{cases} \min \quad \sum_{i=1}^{10} x_i^+ + \sum_{i=1}^{10} x_i^- \\ \textit{subject to} \quad x_i^+ - x_i^- + c_i \le x_j^+ - x_j^- - c_j - 24 \ , \quad j \in E \backslash T^0 ,\ i \in P_j \\ x_i^+ \ge 0, \qquad i = 1, 2, \ldots, 10 \\ 0 \le x_i^- \le c_i ,\ i = 1, 2, \ldots, 10 \end{cases} \tag{23}$$

which is a linear programming model.
Using LiPS software, the following result was obtained

$x_i^+ = \{0, 0, 0, 0, 0, 0, 14, 74, 24, 0, 34\}$

$x_i^- = \{50, 0, 10, 0, 10, 0, 0, 0, 0, 0\}$

$x_i = \{50, 50, 50, 130, 130, 74, 154, 74, 160, 154\}$

The total modification is 216.

5. Conclusions

In this paper, two special types of models for IMST problem were considered with uncertain edge weights and then transformed to the corresponding crisp equivalent models. In the first case, the edge weights were taken as rough variables. The mean and standard deviation of each rough variable were obtained. Then the rough variables were approximated by the uncertain normal variables with the mean and standard deviation of the rough variables. The uncertain inverse minimum spanning tree problem was reduced to a deterministic linear programming model, which can be solved by any standard method. In this case, the rough variables, deviations from x_i are symmetrical. In a case of non-symmetric deviation, the variance of the rough variable shall not be independent of x_i, but rather a linear expression in x_i. So, the standard deviation of the rough variable shall be of the form $\sqrt{Ax_i + B}$ for some constants A and B. So, the model can be transformed to a non-linear programming problem which needs further investigation. In the second case, using trust distribution, the rough IMST was converted to a crisp equivalent model as a linear programming model and solved using a standard method. Two alternative ways were presented to solve the IMST problem. Here we have taken a particular type of rough variable of the form [a, b, c, d], while by taking other forms of rough variables different models of IMST can be formed. IMST with uncertain edge weights could be solved using some other methods like Genetic Algorithm, and could be studied in fuzzy graph, rough fuzzy graph etc.

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