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Original Article

Asymptotic approximation of the norms of monomials in weighted Segal-Bargmann spaces

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Abstract

We study radial weighted Segal-Bargmann spaces

$$H_0 := \left\{ f : \mathbb{C} \to \mathbb{C} \,\middle|\, \|f\|_0^2 = \frac{1}{\pi}\int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} dz < \infty \right\},$$

$$H_1 := \left\{ f : \mathbb{C} \to \mathbb{C} \,\middle|\, \|f\|_1^2 = \frac{1}{\pi}\int_{\mathbb{C}} |f(z)|^2 e^{|z|} e^{-|z|^2} dz < \infty \right\},$$

$$H_{-1} := \left\{ f : \mathbb{C} \to \mathbb{C} \,\middle|\, \|f\|_{-1}^2 = \frac{1}{\pi}\int_{\mathbb{C}} |f(z)|^2 e^{-|z|} e^{-|z|^2} dz < \infty \right\}$$

and investigate the norms of monomials in these spaces. It is well-known that $\|z^k\|_0^2 = k!$. However, we cannot find in closed form the norms $\|z^k\|_1$ and $\|z^k\|_{-1}$. The purpose of this work is to establish an upper bound for $\dfrac{\|z^k\|_1^2 \|z^k\|_{-1}^2}{\|z^k\|_0^4}$.

Keywords: weighted Segal-Bargmann, asymptotic, norm

1. Introduction

The Segal-Bargmann space (also called a Fock space) is the holomorphic function space $HL^2(\mathbb{C}, \mu)$ where $\mu(z) = \dfrac{1}{\pi} e^{-|z|^2}$. It is a Hilbert space of holomorphic functions on $\mathbb{C}$ with inner product given by

$$\langle f, g \rangle = \int_{\mathbb{C}} f(z)\overline{g(z)}\mu(z)\,dz .$$

See Bargmann (1961), Hall (2000), Le (2017), and Soltani (2006). The norm of Z^k in this space can be calculated using polar coordinates as follows:

$$\|z^k\|^2 = \int_{\mathbb{C}} |z^k|^2 \mu(z)\,dz = \frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty} r^{2k+1} e^{-r^2} dr\,d\theta = k!.$$

Therefore, the set $\left\{ \dfrac{z^k}{\sqrt{k!}} \right\}_{k=0}^{\infty}$ forms an orthonormal basis for this space (Hall, 2000).

We commonly weight the measure by multiplying with a nonnegative function in weighted Segal-Bargmann

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space or weighted Fock space. However, there are different varieties of these spaces. For example, the author of Soltani (2006) defined and investigated a weighted Fock space associated with the perturbed Dunkl operator. The inner product in this space is given by

$$\langle f,g\rangle_Q = \int_{\mathbb{C}} f_e(z)\overline{g_e(z)}\,dm_\alpha^Q(z) + 2(\alpha+1)\int_{\mathbb{C}} f_o(z)\overline{g_o(z)}|z|^{-2}\,dm_{\alpha+1}^Q(z)$$

where $\alpha > -1/2$, $f_e(z) = \dfrac{f(z)+f(-z)}{2}$, $f_o(z) = \dfrac{f(z)-f(-z)}{2}$ and a measure $dm_\alpha^Q(z)$ associated with a function Q. In Bergman (2017) and Escudero, Haimi, and Romero (2021), a weighted Fock space is defined as $HL^2(\mathbb{C}, e^{\phi(z)})$ for some plurisubharmonic function $\phi(z)$. In (Choe & Nam, 2019), the t-weighted α-Fock space is introduced as a space consisting of all holomorphic functions f on $\mathbb{C}^n$ such that the integral

$$\int_{\mathbb{C}^n} \left| f(z) e^{-\frac{\alpha}{2}|z|^2} \right|^p \frac{1}{(1+|z|)^t}\, dV(z) < \infty$$

where $\alpha > 0, 0 < p < \infty$ and $dV(z)$ is the volume measure on $\mathbb{C}^n$. The radial weighted Segal-Bargmann space is the variant that we employ in this paper. For $h(z) := h(|z|)$, this weighted Segal-Bargmann space consists of all holomorphic functions on $\mathbb{C}$ such that

$$\int_{\mathbb{C}} |f(z)|^2 e^{-h(z)}\, dz < \infty .$$

(See (Baranov, Belov & Borichev, 2018).)

In this paper, we let $\mu(z) = \frac{1}{\pi} e^{-|z|^2}$ and denote the classical Segal-Bargmann space by $H_0 := HL^2(\mathbb{C},\mu) = \left\{ f:\mathbb{C}\to\mathbb{C} \,\middle|\, \|f\|_0^2 = \frac{1}{\pi}\int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} dz < \infty \right\}$.

By multiplying a positive function $\phi(z)$ to the measure $d\mu(z)$, we obtain another holomorphic function space $HL^2(\mathbb{C}, \phi\mu)$. This new space will be referred to as a weighted Segal-Bargmann space. To make use of polar coordinates as we compute the norm $\|z^k\|_0$, one may assume that the function ϕ is rotation invariant as $\phi = \phi(|z|)$. Since the function $\mu(z) = \frac{1}{\pi} e^{-|z|^2}$ depends only on $|z|$, the space $HL^2(\mathbb{C}, \phi\mu)$ is a radial weighted Segal-Bargmann space.

Let $\phi_1 = e^{|z|}$ and $\phi_{-1} = e^{-|z|}$. Then we define the spaces H_1 and H_{-1} as follows.

$$H_1 := HL^2(\mathbb{C},\phi_1\mu) = \left\{ f:\mathbb{C}\to\mathbb{C} \,\middle|\, \|f\|_1^2 = \frac{1}{\pi}\int_{\mathbb{C}} |f(z)|^2 e^{|z|} e^{-|z|^2} dz < \infty \right\},$$

$$H_{-1} := HL^2(\mathbb{C},\phi_{-1}\mu) = \left\{ f:\mathbb{C}\to\mathbb{C} \,\middle|\, \|f\|_{-1}^2 = \frac{1}{\pi}\int_{\mathbb{C}} |f(z)|^2 e^{-|z|} e^{-|z|^2} dz < \infty \right\}.$$

Consider

$$\frac{1}{\pi}\int_{\mathbb{C}} |z^k|^2 e^{a|z|} e^{-|z|^2} dz = \frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty} r^{2k+1} e^{ar-r^2}\, dr\, d\theta$$

where $a = \pm 1$. Now, the integral $\int r^{2k+1} e^{ar-r^2} dr$ no longer is expressible with elementary functions. However, the integral $\int_{\mathbb{C}} |z^k|^2 e^{a|z|} e^{-|z|^2} dz$ is still finite because the term $\mu(z) = e^{-|z|^2}$ dominates all other terms.

Despite the fact that the formula for $\|z^k\|_a^2 := \frac{1}{\pi}\int_{\mathbb{C}} |z^k|^2 e^{a|z|} e^{-|z|^2} dz$ is implicit, the behavior of the growth of $\|z^k\|_a^2$ in terms of k is remarkably similar to that of $\|z^k\|_0^2$. We shall show in Section 2 that the functions $r^{2k+1}e^{-r^2}$, $r^{2k+1}e^{r-r^2}$ and $r^{2k+1}e^{-r-r^2}$ are all concentrated towards the peaks of these functions. As a result, the norms $\|z^k\|_0^2$, $\|z^k\|_1^2$ and $\|z^k\|_{-1}^2$ can be approximated asymptotically by definite integrals.

In Chailuek and Senmoh (2020), the authors show that the boundedness of $\frac{\|z^k\|_\alpha^2 \|z^k\|_\beta^2}{\|z^k\|_\gamma^4}$ plays an important role in a proof of the dual of a generalized Bergman space, $HL^2(\mathrm{B}^d,\alpha)^* = HL^2(\mathrm{B}^d,\beta)$ under the integral pairing

$$\langle f,g\rangle_\gamma = \int_{\mathrm{B}^d} f(z)\overline{g(z)}c_\lambda\left(1-|z|^2\right)^{\lambda-(d+1)}dz$$

for $f \in H\left(B^d,\alpha\right), g \in H\left(B^d,\beta\right)$.

Despite the fact that the formulas for $\|z^k\|_1^2$ and $\|z^k\|_{-1}^2$ are implicit, we will show in Section 3 that $\frac{\|z^k\|_1^2\|z^k\|_{-1}^2}{\|z^k\|_0^4}$ is asymptotically bounded above by a constant.

2. Norms of Monomials in Segal-Bargmann Spaces

In the classical Segal-Bargamann space, the norm of a monomial can be computed explicitly as $\|z^k\|_0^2 = \frac{1}{\pi}\int_0^{2\pi}\int_0^\infty r^{2k+1}e^{-r^2}drd\theta = 2\int_0^\infty r^{2k+1}e^{-r^2}dr = k!$. Consider the graph of $f_k(r) = r^{2k+1}e^{-r^2}$. It resembles a Gaussian-shaped wave function that propagates to the right as k increases. (Figure 1.)

In this section, we will show that the function f_k behaves like a Gaussian-shaped wave function in the sense that it is concentrated towards its peak and likely to have a finite width which is measured from where the function is somehow cut off. Consequently, the integral $\int_0^\infty r^{2k+1}e^{-r^2}dr$ can be estimated by a definite integral $\int_0^\infty r^{2k+1}e^{-r^2}dr \,\mathbb{C} \int_0^{2r_0} r^{2k+1}e^{-r^2}dr$ for some $r_0 > 0$.

As previously stated, explicit formulas for $\|z^k\|_1$ and $\|z^k\|_{-1}$ are unavailable. However, when we compare the graphs of $f_{k,-1}(r) = r^{2k+1}e^{-r-r^2}$

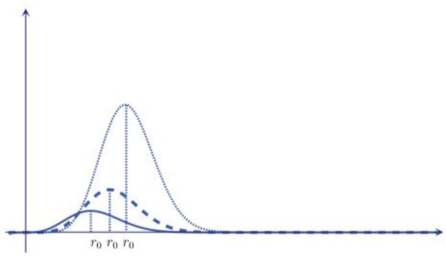

Figure 1. The graphs of $f_k(r) = r^{2k+1}e^{-r^2}$ for different k's.

and $f_{k,1}(r) = r^{2k+1}e^{r-r^2}$ to that of $f_k(r) = r^{2k+1}e^{-r^2}$. We can see that they are similarly concentrated towards their peaks and have finite widths. (Figure 2.)

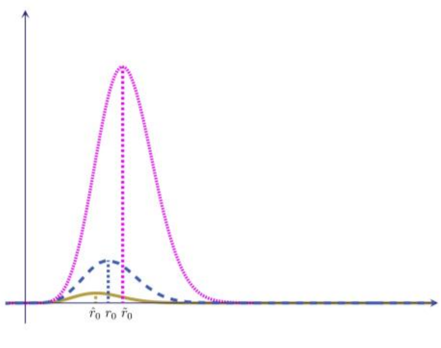

Figure 2. The graphs of $f_{k,-1}(r)$, $f_{k,1}(r)$ and $f_k(r)$.

So, it makes sense to estimate those integrals by definite integrals. Therefore, the goals of this section are to compare $\left\|z^k\right\|_{-1}^2$ and $\left\|z^k\right\|_1^2$ with $\left\|z^k\right\|_0^2$ as we obtain

$$\frac{\left\|z^k\right\|_{-1}^2}{\left\|z^k\right\|_0^2} \sim \frac{\int_0^{2\hat{r}_0} r^{2k+1}e^{-r-r^2}dr}{\int_0^{2r_0} r^{2k+1}e^{-r^2}dr} \text{ and } \frac{\left\|z^k\right\|_1^2}{\left\|z^k\right\|_0^2} \sim \frac{\int_0^{2\tilde{r}_0} r^{2k+1}e^{r-r^2}dr}{\int_0^{2r_0} r^{2k+1}e^{-r^2}dr}$$

for some $\hat{r}_0, \tilde{r}_0 > 0$. We begin by generating some relevant identities as follows.

Lemma 2.1 $\left\|z^k\right\|_0^2 = k!$ where *k* is a nonnegative integer.

Proof. We compute $\left\|z^k\right\|_0^2$ by induction on *k*. For *k* = 0,

$$\int_0^\infty re^{-r^2}dr = -\frac{1}{2}\lim_{t\to\infty} e^{-r^2}\Big|_0^t = \frac{1}{2}.$$

For $k \geq 1$, integrating by parts gives

$$\int_0^\infty r^{2k+1}e^{-r^2}dr = -\int_0^\infty \left(2kr^{2k-1}\right)\left(-\frac{e^{-r^2}}{2}\right)dr$$
$$= k\int_0^\infty r^{2(k-1)+1}e^{-r^2}dr.$$

Therefore, $\int_0^\infty r^{2k+1}e^{-r^2}dr = \frac{k!}{2}$ and hence $\left\|z^k\right\|_0^2 = k!$.

Lemma 2.2 For a nonnegative integer *n* and $a, b > 0$.

$$\int_0^b x^n e^{-ax}dx = \frac{n!}{a^{n+1}}\left(1 - e^{-ab}\sum_{i=0}^n \frac{(ab)^i}{i!}\right). \tag{2.1}$$

Proof. Integration by parts gives

$$\int_0^b x^n e^{-ax}dx = -\frac{x^n e^{-ax}}{a} - \frac{nx^{n-1}e^{-ax}}{a^2} - \cdots - \frac{n!xe^{-ax}}{a^n} - \frac{n!e^{-ax}}{a^{n+1}}\Big|_0^b$$
$$= \frac{n!}{a^{n+1}}\left(1 - e^{-ab}\sum_{i=0}^n \frac{(ab)^i}{i!}\right).$$

Lemma 2.3 For $r_0 = \sqrt{\frac{2k+1}{2}}$, $\lim_{k\to\infty} e^{-4r_0^2}\sum_{i=0}^k \frac{\left(4r_0^2\right)^i}{i!} = 0$.

Proof. For $i = 0,1,2,\ldots,k$, we have $i+1 < 4k+2$ for all positive integer *k*.

Thus $\frac{(4r_0^2)^i}{i!} < \frac{(4r_0^2)^{i+1}}{(i+1)!}$ and hence

$$0 < e^{-4r_0^2}\sum_{i=0}^k \frac{\left(4r_0^2\right)^i}{i!} < e^{-4r_0^2}(k+1)\frac{\left(4r_0^2\right)^k}{k!} = e^{-(4k+2)}(k+1)\frac{(4k+2)^k}{k!}.$$

It's not difficult to understand that the last quantity tends to zero.

Next, we will show that $\left\|z^k\right\|_0^2$ is asymptotically equal to a definite integral as follows.

Proposition 2.4 Let $k = 0,1,2,3,\ldots$ and $r_0 = \sqrt{\frac{2k+1}{2}}$ be the critical point of $f_k(r) = r^{2k+1}e^{-r^2}$. Then $\left\|z^k\right\|_0^2 \sim 2\int_0^{2r_0} r^{2k+1}e^{-r^2}dr$.

Proof. From Lemma 2.1, we obtain

$$\int_0^\infty r^{2k+1}e^{-r^2}dr = \frac{k!}{2}. \tag{2.2}$$

Substitute $n = k, a = 1$, and $b = 4r_0^2$ in the equation (2.1), to obtain

$$\int_0^{2r_0} r^{2k+1}e^{-r^2}dr=\frac{1}{2}\int_0^{4r_0^2} s^k e^{-s}ds=\frac{k!}{2}\left(1-e^{-4r_0^2}\sum_{i=0}^{k}\frac{\left(4r_0^2\right)^i}{i!}\right). \quad (2.3)$$

From equations (2.2) and (2.3), we obtain

$$\frac{\int_0^{2r_0} r^{2k+1}e^{-r^2}dr}{\int_0^{\infty} r^{2k+1}e^{-r^2}dr}=1-e^{-4r_0^2}\sum_{i=0}^{k}\frac{\left(4r_0^2\right)^i}{i!}.$$

From Lemma 2.3, we obtain

$$\lim_{k\to\infty} e^{-4r_0^2}\sum_{i=0}^{k}\frac{\left(4r_0^2\right)^i}{i!}=0.$$

Thus,

$$\lim_{k\to\infty}\frac{\int_0^{2r_0} r^{2k+1}e^{-r^2}dr}{\int_0^{\infty} r^{2k+1}e^{-r^2}dr}=\lim_{k\to\infty}\left(1-e^{-4r_0^2}\sum_{i=0}^{k}\frac{\left(4r_0^2\right)^i}{i!}\right)=1.$$

Therefore, $\int_0^{\infty} r^{2k+1}e^{-r^2}dr \sim \int_0^{2r_0} r^{2k+1}e^{-r^2}dr$. Hence, $\left\|z^k\right\|_0^2=2\int_0^{\infty} r^{2k+1}e^{-r^2}dr \sim 2\int_0^{2r_0} r^{2k+1}e^{-r^2}dr$.

Recall that $\left\|z^k\right\|_1^2=2\int_0^{\infty} r^{2k+1}e^{r-r^2}dr$ and $\left\|z^k\right\|_{-1}^2=2\int_0^{\infty} r^{2k+1}e^{-r-r^2}dr$.

Although we can derive the closed form of the integral $\int_0^{\infty} r^{2k+1}e^{-r^2}dr$ using integration by substitution and induction, there is no elementary function whose derivative is $r^{2k+1}e^{-r-r^2}$ or $r^{2k+1}e^{r-r^2}$. The functions $r^{2k+1}e^{-r-r^2}$ or $r^{2k+1}e^{r-r^2}$ behave similarly to the function $f_k(r)=r^{2k+1}e^{-r^2}$ when k is fixed.

As a result, we focus our attention on the asymptotic approximation of $\left\|z^k\right\|_1^2 \big/ \left\|z^k\right\|_0^2$ and $\left\|z^k\right\|_{-1}^2 \big/ \left\|z^k\right\|_0^2$.

Proposition 2.5 Let k = 0,1,2,3, … . Then

$$\frac{\left\|z^k\right\|_{-1}^2}{\left\|z^k\right\|_0^2}\sim\frac{\int_0^{2\hat{r}_0} r^{2k+1}e^{-r-r^2}dr}{\int_0^{2r_0} r^{2k+1}e^{-r^2}dr} \text{ and } \frac{\left\|z^k\right\|_1^2}{\left\|z^k\right\|_0^2}\sim\frac{\int_0^{2\tilde{r}_0} r^{2k+1}e^{r-r^2}dr}{\int_0^{2r_0} r^{2k+1}e^{-r^2}dr}$$

where $r_0=\sqrt{\frac{2k+1}{2}}$, $\hat{r}_0=\frac{-1+\sqrt{16k+9}}{4}$ and $\tilde{r}_0=\frac{1+\sqrt{16k+9}}{4}$ are the critical points of $f_k(r)=r^{2k+1}e^{-r^2}$, $f_{k,-1}(r)=r^{2k+1}e^{-r-r^2}$ and $f_{k,1}(r)=r^{2k+1}e^{r-r^2}$, respectively.

Proof. Consider

$$\frac{\left\|z^k\right\|_{-1}^2}{\left\|z^k\right\|_0^2}\sim\frac{\int_0^{2\hat{r}_0} r^{2k+1}e^{-r-r^2}dr}{\int_0^{2r_0} r^{2k+1}e^{-r^2}dr}+\frac{\int_{2\hat{r}_0}^{\infty} r^{2k+1}e^{-r-r^2}dr}{\int_0^{2r_0} r^{2k+1}e^{-r^2}dr}.$$

Since $0<r^{2k+1}e^{-r-r^2}<r^{2k+1}e^{-r^2}$,

$$\frac{\int_{2\hat{r}_0}^{\infty} r^{2k+1}e^{-r-r^2}dr}{\int_0^{2r_0} r^{2k+1}e^{-r^2}dr}\le\frac{\int_{2\hat{r}_0}^{\infty} r^{2k+1}e^{-r^2}dr}{\int_0^{2r_0} r^{2k+1}e^{-r^2}dr}=\frac{\int_0^{\infty} r^{2k+1}e^{-r^2}dr-\int_0^{2\hat{r}_0} r^{2k+1}e^{-r^2}dr}{\int_0^{2r_0} r^{2k+1}e^{-r^2}dr}.$$

By using integration by substitution and substituting $n = k$, $a = 1$, and $b = 4r_0^2$ into the equation (2.1), we obtain

$$\int_0^{2\hat{r}_0} r^{2k+1} e^{-r^2} dr = \frac{k!}{2}\left(1 - e^{-4\hat{r}_0^2} \sum_{i=0}^{k} \frac{\left(4\hat{r}_0^2\right)^i}{i!}\right). \tag{2.4}$$

From equations (2.2), (2.3) and (2.4), we obtain

$$\lim_{k\to\infty} \frac{\int_{2\tilde{r}_0}^{\infty} r^{2k+1} e^{-r-r^2} dr}{\int_0^{2r_0} r^{2k+1} e^{-r^2} dr} = 0.$$

Therefore,

$$\frac{\left\|z^k\right\|_{-1}^2}{\left\|z^k\right\|_0^2} \sim \frac{\int_0^{2\hat{r}_0} r^{2k+1} e^{-r-r^2} dr}{\int_0^{2r_0} r^{2k+1} e^{-r^2} dr}.$$

Now, consider

$$\frac{\left\|z^k\right\|_1^2}{\left\|z^k\right\|_0^2} \sim \frac{\int_0^{2\tilde{r}_0} r^{2k+1} e^{r-r^2} dr}{\int_0^{2r_0} r^{2k+1} e^{-r^2} dr} + \frac{\int_{2\tilde{r}_0}^{\infty} r^{2k+1} e^{r-r^2} dr}{\int_0^{2r_0} r^{2k+1} e^{-r^2} dr}.$$

Let r be an element in an interval $(2\tilde{r}_0, \infty)$. The function e/e^r is decreasing and $e/e^r \to 0$ as $r \to \infty$; on the other hand, the function $(r-1)^{2k+1}/r^{2k+1}$ is increasing and $(r-1)^{2k+1}/r^{2k+1} \to 1$ as $r \to \infty$. Consider $r = 2r_0$. We see that $\frac{e}{e^{2r_0}} \le \left(\frac{2r_0 - 1}{2r_0}\right)^{2k+1}$ for all k. Thus, we obtain $\frac{e}{e^r} \le \left(\frac{r-1}{r}\right)^{2k+1}$ and hence $r^{2k+1} \le (r-1)^{2k+1} e^{r-1}$ for all $r \ge 2\tilde{r}_0$. Therefore,

$$\int_{2\tilde{r}_0}^{\infty} r^{2k+1} e^{r-r^2} dr \le \int_{2\tilde{r}_0}^{\infty} (r-1)^{2k+1} e^{-(r-1)^2} dr.$$

By using integration by substitution and equations (2.1) and (2.2), we have

$$\int_{2\tilde{r}_0}^{\infty} (r-1)^{2k+1} e^{-(r-1)^2} dr = \frac{k!}{2} - \frac{k!}{2}\left(1 - e^{-(2\tilde{r}_0-1)^2} \sum_{i=0}^{k} \frac{(2\tilde{r}_0 - 1)^{2i}}{i!}\right). \tag{2.5}$$

From equations (2.3) and (2.5), we obtain

$$\lim_{k\to\infty} \frac{\int_{2\tilde{r}_0}^{\infty} (r-1)^{2k+1} e^{-(r-1)^2} dr}{\int_0^{2r_0} r^{2k+1} e^{-r^2} dr} = 0.$$

Therefore, we obtain $\frac{\left\|z^k\right\|_1^2}{\left\|z^k\right\|_0^2} \sim \frac{\int_0^{2\tilde{r}_0} r^{2k+1} e^{r-r^2} dr}{\int_0^{2r_0} r^{2k+1} e^{-r^2} dr}.$

3. The Boundedness of $\frac{\left\|z^k\right\|_1^2 \left\|z^k\right\|_{-1}^2}{\left\|z^k\right\|_0^4}$

It is easy to see that $\left\|z^k\right\|_{-1}^2 \le \left\|z^k\right\|_0^2 \le \left\|z^k\right\|_1^2$. This implies that the ratio $\left\|z^k\right\|_{-1}^2 / \left\|z^k\right\|_0^2$ may decrease, whilst the ratio $\left\|z^k\right\|_1^2 / \left\|z^k\right\|_0^2$ may increase. We shall demonstrate in this section that these two quantities are mutually compensated resulting in the boundedness of $\frac{\|z^k\|_1^2 \|z^k\|_{-1}^2}{\|z^k\|_0^4}$. In addition, the upper bound is involved in the peaks of f_k, $f_{k,-1}$ and $f_{k,1}$.

Since $\hat{r}_0 \sim r_0 \sim \tilde{r}_0$ and $|\tilde{r}_0 - r_0| \sim |r_0 - \hat{r}_0|$, it should come as no surprise that the values $f_k(r_0)$, $f_{k,-1}(\hat{r}_0)$ and $f_{k,1}(\tilde{r}_0)$ are somehow offset.

Theorem 3.1 Let k = 0,1,2,3,… . Then

$$\frac{\|z^k\|_1^2 \|z^k\|_{-1}^2}{\|z^k\|_0^4} \sim e^{\frac{1}{4}}.$$

Proof. From the previous section, we have

$$\frac{\|z^k\|_1^2 \|z^k\|_{-1}^2}{\|z^k\|_0^4} \sim \frac{\int_0^{2\tilde{r}_0} r^{2k+1} e^{r-r^2} dr \int_0^{2\hat{r}_0} r^{2k+1} e^{-r-r^2} dr}{\left(\int_0^{2r_0} r^{2k+1} e^{-r^2} dr\right)^2}. \tag{2.6}$$

First, we consider the definite integral

$$\int_0^{2r_0} r^{2k+1} e^{-r^2} dr = \int_0^{2r_0} e^{-r^2+(2k+1)\ln r} dr = \int_0^{2r_0} e^{f(r)} dr$$

where $f(r) = -r^2 + (2k+1)\ln r$.

The Taylor series expansion of $f(r)$ about a point $r = r_0$ is given by

$$f(r) = \sum_{n=0}^{\infty} \frac{f^n(r_0)}{n!}(r-r_0)^n$$

with the interval of convergence $(0, 2r_0)$. Thus,

$$\int_0^{2r_0} e^{f(r)} dr = \int_0^{2r_0} e^{f(r_0)+f'(r_0)(r-r_0)+\frac{f''(r_0)(r-r_0)^2}{2!}+\sum_{n=3}^{\infty}\frac{f^n(r_0)}{n!}(r-r_0)^n} dr.$$

We have $f'(r_0) = 0$ and $f''(r_0) = -4$. If we consider $k \to \infty$, then $f^m(r_0) \to 0$ for all $m \geq 3$. Therefore,

$$\int_0^{2r_0} e^{f(r)} dr = e^{f(r_0)} \int_0^{2r_0} e^{-2(r-r_0)^2} dr = e^{f(r_0)} \int_{-r_0}^{r_0} e^{-2u^2} du \tag{2.7}$$

where $u = r - r_0$.

Next, we consider the definite integral

$$\int_0^{2\tilde{r}_0} r^{2k+1} e^{r-r^2} dr = \int_0^{2\tilde{r}_0} e^{r-r^2+(2k+1)\ln r} dr = \int_0^{2\tilde{r}_0} e^{\tilde{f}(r)} dr$$

where $\tilde{f}(r) = r - r^2 + (2k+1)\ln r$.

Similarly, we have

$$\int_0^{2\tilde{r}_0} e^{\tilde{f}(r)} dr \sim e^{\tilde{f}(\tilde{r}_0)} \int_0^{2\tilde{r}_0} e^{-2(r-\tilde{r}_0)^2} dr = e^{\tilde{f}(\tilde{r}_0)} \int_{-\tilde{r}_0}^{\tilde{r}_0} e^{-2\tilde{u}^2} d\tilde{u} \tag{2.8}$$

where $\tilde{u} = r - \tilde{r}_0$ and

$$\int_0^{2\hat{r}_0} e^{\hat{f}(r)} dr \sim e^{\hat{f}(\hat{r}_0)} \int_0^{2\hat{r}_0} e^{-2(r-\hat{r}_0)^2} dr = e^{\hat{f}(\hat{r}_0)} \int_{-\hat{r}_0}^{\hat{r}_0} e^{-2\hat{u}^2} d\tilde{u} \tag{2.9}$$

where $\hat{f}(r) = -r - r^2 + (2k+1)\ln r$ and $\hat{u} = r - \hat{r}_0$.

Observe that $r_0 \sim \tilde{r}_0 \sim \hat{r}_0$ as $k \to \infty$. Thus,

$$\int_{-r_0}^{r_0} e^{-2u^2} du \sim \int_{-\tilde{r}_0}^{\tilde{r}_0} e^{-2\tilde{u}^2} d\tilde{u} \sim \int_{-\hat{r}_0}^{\hat{r}_0} e^{-2\hat{u}^2} d\hat{u}.$$

Therefore,

$$\frac{\|z^k\|_1^2 \|z^k\|_{-1}^2}{\|z^k\|_0^4} \sim e^{\tilde{f}(\tilde{r}_0)+\hat{f}(\hat{r}_0)-2f(r_0)}.$$

Next, we compute

$$2f(r_0) = -2k - 1 + (2k+1)\ln\left(k+\frac{1}{2}\right)$$

and

$$\hat{f}(\hat{r}_0) + \tilde{f}(\tilde{r}_0) = \frac{1}{4} - 2k - 1 + (2k+1)\ln\left(k+\frac{1}{2}\right).$$

Therefore $\hat{f}(\hat{r}_0) + \tilde{f}(\tilde{r}_0) = 2f(r_0) + \frac{1}{4}$. This yields

$$\frac{\|z^k\|_1^2 \|z^k\|_{-1}^2}{\|z^k\|_0^4} \sim e^{\hat{f}(\hat{r}_0)+\tilde{f}(\tilde{r}_0)-2f(r_0)} = e^{\frac{1}{4}}.$$

Finally, we obtain that $\frac{\|z^k\|_1^2 \|z^k\|_{-1}^2}{\|z^k\|_0^4}$ is asymptotically less than a constant $e^{\frac{1}{4}}$.

We notice that the estimates in (2.7), (2.8) and (2.9) look similar to the integral asymptotic $\int_a^b f(t) e^{-\lambda g(t)} dt \sim e^{-\lambda g(c)} f(c) \sqrt{\frac{2\pi}{\lambda g''(c)}}$ as $\lambda \to \infty$ where c represents the critical point of g. Using Taylor's expansion and Laplace's method, the integral is involved in the value at the critical point.

4. Conclusions

In this paper, we obtained that $\frac{\|z^k\|_1^2 \|z^k\|_{-1}^2}{\|z^k\|_0^4}$ is asymptotically less than the constant $e^{\frac{1}{4}}$. This implies that $\frac{\|z^k\|_1^2 \|z^k\|_{-1}^2}{\|z^k\|_0^4}$ is bounded and independent of k. Future research could use the boundedness of $\frac{\|z^k\|_1^2 \|z^k\|_{-1}^2}{\|z^k\|_0^4}$ to describe the dual of reciprocal weighted Segal-Bargmann spaces, $H_1{}^* = H_{-1}$ under the integral pairing

$$\langle F, S \rangle_0 = \frac{1}{\pi} \int_{\mathbb{C}} F(z) \overline{S(z)} e^{-|z|^2} dz$$

where $F \in H_1$ and $S \in H_{-1}$.

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