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Original Article

Different types of primary ideals of near-rings and their graphs

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Abstract

In the last few decades, studies of graph theory based on algebraic structures have been pursued. Many studies have inspected graphs via groups, rings, near-rings, and seminear-rings, and vice versa. In our study, first we introduce different type of *υ*-primary ideals (*υ* = 0, 1, 2, 3, *c*), which generalize *υ*-prime ideals (*υ* = 0, 1, 2, 3, *c*) in near-rings. Then, we provide some important characterizations of these newly initiated ideals. In addition, we explore some relationships among these ideals. Throughout, we furnish our results with appropriate examples. Finally, as an application, we provide characterizations of different graphs associated with these *υ*-primary ideals (*υ* = 0, 1, 2, 3, *c*) in near-rings.

Keywords: near-rings, prime ideal, almost prime ideal, primary ideal, graph

1. Introduction

A (right) near-ring is an algebraic structure N with the binary operations "+" and "·" where N is a group under "+", semigroup under "·", and N satisfies the (right) distributive law, i.e., for any $a, b, c \in N$; $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$. Similarly, one can define a left near-ring by requiring the left distributive law. We refer to Pilz (2011) for the fundamental concepts and notions of near-rings. In this note, we deal with a right near-ring. A subset I of a near-ring N is said to be a right ideal of N if: (i) $(I, +) \trianglelefteq (N, +)$, and (ii) $IN \subseteq I$. Similarly, I is said to be a left ideal if (i) $(I, +) \trianglelefteq (N, +)$, (ii) $a_1(a_2 + i) - a_1a_2 \in I$, for all $a_1, a_2 \in N$ and $i \in I$. If I is a left as well as right ideal of a near-ring, then I is called simply an ideal of the near-ring. An ideal I is said to be a prime ideal of N if $I_1I_2 \subseteq I \Rightarrow I_1 \subseteq I$ or $I_2 \subseteq I$ where I_1 and I_2 are both ideals of N. Various types of prime ideals in literature have been discussed in Birkenmeier, Heatherly, and Lee (1993), Fröhlich (1958), Holcombe (1970), Ramakotaiah and Rao (1979). An almost prime ideal has been endorsed by B. Elavarasan (Elavarasan, 2011) in near-rings. I is called an almost prime ideal if $[I_1I_2 \subseteq I$ and $I_1I_2 \nsubseteq I^2] \Longrightarrow [I_1 \subseteq I$ or $I_2 \subseteq I]$, where I_1 and I_2 are ideals of N. Further, the author recognized some relationships involving almost prime and prime ideals as well, in Elavarasan (2011). Recently, almost prime ideal has been introduced in gamma near-ring by Khan, Muhammad, Taouti, and Maki (2018). Notions of 0-(1-2)-prime ideals have been defined in Birkenmeier, Heatherly, and Lee (1993), Holcombe (1970), Ramakotaiah and Rao (1979). Subsequently, Ramakotaiah and Rao (1979) introduced the notions of 0-prime, 1-prime and 2-prime ideals of a near-ring. Following Birkenmeier, Heatherly, and Lee (1993), I is said to be a type-zero or simply a prime ideal of N if $I_1I_2 \subseteq I \Rightarrow I_1 \subseteq I$ or $I_2 \subseteq I$. Further, I is called 3-prime ideal of N when if $aNb \subseteq I$ then $a \in I$ or $b \in I$ (Groenewald, 1991).

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Likewise, I is called 2-prime ideal if for two subgroup K_1, K_2 of $(N, +)$ it holds that $K_1K_2 \subseteq I \Rightarrow K_1 \subseteq I$ or $K_2 \subseteq I$. It is well-known that 3-prime implies 2-prime implies 1-prime implies 0-prime, but the converse doesn't hold for any of these implications. Recently, *P*-ideals and their *P*-properties in near-rings have been introduced in Atagun, Kamaci, Tastekin, and Sezgin (2019). On the other hand, few concepts of near-rings have been shifted towards seminear-rings in Koppula, Kedukodi, and Kuncham (2020). A. Ali introduced commutativity of a 3-prime near-ring on Jordan ideals in Ali (2020). G. Wendt discussed the primeness and primitivity of near-rings in Wendt (2021). Further, T. Gaketem introduced some applications of ideals of nLA-rings in Gaketem (2022). Recently, completely 2-absorbing ideals of N-groups have been discussed in Sahoo, Shetty, Groenewald, Harikrishnan, and Kuncham (2021).

Graph theory is an important subject, able to convert algebraic structures into graphs for improved understanding. We refer to Godsil and Royle (2001), Bondy and Murty (1976) for the basic concepts of graph theory. In Beck (1988) the author introduced the graph of a commutative ring by considering the elements of ring as vertices in the graph, and there exists an edge between vertices *x*, *y* if *x*, *y* = 0. Further, Anderson and Livingston (1999) associate a graph to a commutative ring by using the concepts of zero divisors graph. In Lipkovski (2012) the author associates a digraph with a commutative ring and also discussed some of the properties related to degree and loops of the digraph. In Hausken and Skinner (2013) the authors introduced a digraph for commutative rings and also discussed some properties of digraph of commutative ring, which gives information about the ring. Moreover, Bhavanari, Kunham, and Kedukodi (2010) define the graph of an ideal of near-ring and also introduce the terms strong vertex cut and ideal symmetric graph. Prohelika Das (Das, 2016) defined the diameter, girth and coloring of the strong zero-divisor graph of near-rings. Recently, graph of prime intersection of ideals in ring has been introduced in Rajkhowa and Saikia (2020).

In this study, we define the notions of υ-primary ideals (υ = 0, 1, 2, 3, *c*) which are generalizations of υ-prime ideals (υ = 0, 1, 2, 3, *c*) in near-rings. Further, we investigate that 0-prime ideal showing it is always 0-primary, but the converse is not true. We also establish the relations among different υ-primary ideals (υ = 0, 1, 2, 3, *c*) as well as with υ-prime ideals and verify these relations by suitable examples. Furthermore, several characterizations are obtained and supported by suitable examples. Finally, we define the graphs of υ-primary ideals (υ = 0, 1, 2, 3, *c*) in near-ring and verify these definitions of υ-primary ideals (υ = 0, 1, 2, 3, *c*) by using the concepts of subgraphs.

2. Primary Ideals in Near-Rings

In this section, we introduce and discuss different types of υ-primary ideals (υ = 0, 1, 2, 3, *c*) of near-rings. We also investigate some relationships among them.

Definition 2.1 A proper ideal *P* of near-ring *N* is called 0-primary if for all ideals I_1, I_2 of *N* it holds that $I_1I_2 \subseteq P \Rightarrow I_1 \subseteq P$ or $I_2{}^n \subseteq P$ for some $n \in Z^+$.

Example 2.2 Let *N* = {0, 1, 2, 3, 4, 5, 6, 7} be a right near-ring with the two operations defined in the table set 1.

Table 1. Set 1

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

·	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	1	0	1
2	0	0	2	2	0	0	2	2
3	0	0	2	2	0	1	2	3
4	0	1	0	1	4	4	4	4
5	0	1	0	1	4	5	4	4
6	0	1	2	3	4	4	6	6
7	0	1	2	3	4	5	6	7

Here, {0}, {0, 1}, {0, 2}, {0, 1, 2, 3} and *N* are all ideals of *N*. If we choose, *P* = {0, 2} then for all I_1, I_2 of *N*, $I_1I_2 \subseteq P$ implies $I_1 \subseteq P$ or $I_2^n \subseteq P$. For clarification we can take $I_1 = \{0, 1, 2, 3\}$ and $I_2 = \{0, 1\}$ as ideals of *N* such that $I_1I_2 = \{0\} \subseteq P$ implies $I_2^2 \subseteq P$. This implies that *P* is 0-primary ideal of *N*. Similarly, one can justify for the remaining ideals.

Proposition 2.3 Let I be an ideal of a zero-symmetric near-ring *N*. Then, I is a 0-primary ideal if and only if every zero-divisor in N/I is nilpotent.

Proof. Let I be a 0-primary ideal and consider N/I being non-trivial. Let $n + I \in N/I$ be a zero-divisor and $n_1 \in N/I$. Let $n_1n \ +I = (n_1 + I)(n + I) = 0 + I \Rightarrow \ n_1n \in I, \ n_1 \notin I \Rightarrow n^k \in I$ for some $k \in Z^+$. Hence

$(n+I)^k = n^k + I = 0 + I \Rightarrow n + I$ is nilpotent. Conversely, let N/I be non-trivial and every nonzero zero-divisor in N/I is nilpotent. Since $I \neq N$, let $n_1, n_2 \in N$ such that $n_1 \cdot n_2 \in I$, then each $n_1 \in I$ or $n_1 \notin I$, suppose $n_1 \notin I$ then consider $(n_2 + I)(n_1 + I) = n_2 \cdot n_1 + I = 0 + I \Rightarrow n_2 \cdot n_1 = 0$, so $n_2 + I$ is a zero-divisor and there exists $k > 0$ such that $n_2^k + I = (n_2 + I)^k = I \Rightarrow n_2^k \in I$, hence I is a 0-primary ideal.

Example 2.4 Suppose *N* = {0, 1, 2, 3} is the zero-symmetric near-ring under the addition and multiplication defined in tables set 2.

Table 2. Set 2

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

·	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	0	0
3	0	1	0	1

Clearly, *P* = {0, 1} is 0-primary ideal and the quotient *N/P* = {0 + *P*, 2 + *P*} along with operations given in tables set 3.

Table 3. Set 3

+	0 + *P*	2 + *P*
0 + *P*	0 + *P*	2 + *P*
2 + *P*	2 + *P*	0 + *P*

·	0 + *P*	2 + *P*
0 + *P*	0 + *P*	0 + *P*
2 + *P*	0 + *P*	0 + *P*

Here the zero divisors of *N/P* are 0 + *P* and 2 + *P*, which are nilpotent.

Intersection of any two 0-primary ideals of a near-ring is a 0-primary ideal, and we provide an example of this fact below.

Example 2.5 Suppose *N* = {0, 1, 2, 3} is a commutative near-ring with "+"and " ·" defined in tables set 2 of Example 2.4. Let us consider the 0-primary ideals $P_1 = \{0, 1\}$ and $P_2 = \{0, 2\}$ of the near-ring *N*. Now $P_1 \cap P_2 = \{0\}$ is also a 0-primary ideal of *N*.

Proposition 2.6 Each 0-prime is a 0-primary ideal of *N*.

Proof. Let *P* be a 0-prime ideal of a near-ring *N*. Then, $ab \in P$ implies $a \in P$ or $b \in P$, for all $a, b \in N$ while considering *n* = 1, and the result follows.

Remark 2.7 Each maximal ideal of *N* is 0-prime and hence a 0-primary ideal. This implies that a maximal ideal is a 0-primary.

Definition 2.8 A proper ideal I of a near-ring *N* is said to be a semi-primary ideal, if for ideal J of N, $J^2 \subseteq I \Rightarrow J \subseteq I$.

It is well known that the intersection of prime ideals in a near-ring is also a semi-prime ideal. We also know that the intersection of minimal prime ideals of *N* is a semi-prime ideal of *N* such that the ideal I can be written as the intersection of all prime ideals containing I. However, the intersection of two primary ideals need not be a semi-primary ideal, for instance see Example 2.5, i.e., $I = \{0\}$ is the intersection of 0-primary ideals $\{0, 1\}$ and $\{0, 2\}$, but I is not a semi-primary ideal, i.e., $P_2^2 \subseteq \{0\} = I$, but $P_2 \nsubseteq I$.

Remark 2.9 Every 0-primary ideal is not necessarily a semi-primary ideal.

To verify above remark we refer to Example 2.5, in which {0, 2} is 0-primary ideal but not a semi-primary ideal.

Definition 2.10 A proper ideal *P* of *N* is called 1-primary ideal, if for all I_1, I_2 that are right ideals of *N*, it holds that $I_1 I_2 \subseteq P \Rightarrow I_1 \subseteq P$ or $I_2^n \subseteq P$ for some $n \in \mathbb{Z}^+$.

Example 2.11 Consider *N* = {0, 1, 2, 3, 4, 5, 6, 7}that is a right near-ring with addition and multiplication defined in tables set 4.

Table 4. Set 4

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	0
2	2	3	4	5	6	7	0	1
3	3	4	5	6	7	0	1	2
4	4	5	6	7	0	1	2	3
5	5	6	7	0	1	2	3	4
6	6	7	0	1	2	3	4	5
7	7	0	1	2	3	4	5	6

·	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	0	1	0	1	1	0
2	0	2	0	2	0	2	2	0
3	0	3	0	3	0	3	3	0
4	4	4	4	4	4	4	4	4
5	4	5	4	5	4	5	5	4
6	4	6	4	6	4	6	6	4
7	4	7	4	7	4	7	7	4

Here{0}, {0, 2, 4, 6}, {0, 2} and *N* are all ideals of *N*. If we choose, *P* = {0, 4}, then for all right ideals I_1, I_2 of *N*, it holds that $I_1 I_2 \subseteq P$ implies $I_1 \subseteq P$ or $I_2^n \subseteq P$ for some $n \in Z^+$. To verify this let us consider $I_1 = \{0, 2, 4, 6\}$ and $I_2 = \{0, 2\}$ are non-travail right ideals of *N*, such that $I_1 I_2 = \{0, 4\} \subseteq P$ implies $I_1 \nsubseteq P$ but $I_2^2 = \{0\} \subseteq P$. As a result, *P* is a 1-primary ideal of *N*.

Definition 2.12 A proper ideal *P* of *N* is called 2-primary ideal if for all *N*-subgroups I_1 and I_2, it holds that $I_1 I_2 \subseteq P \Rightarrow I_1 \subseteq P$ or $I_2^n \subseteq P$ for some $n \in Z^+$.

Proposition 2.13 Let *N* be a near-ring. Then the given statements are equivalent.
(i) *P* is a 2-primary ideal of *N*.
(ii) If *A* is an *N*-subgroup and *B* is an ideal of *N*, then $AB \subseteq P$ implies $A \subseteq P$ or $B^k \subseteq P$ where $k \in Z^+$.

Proof. $(i) \Rightarrow (ii)$ If *P* is a 2-primary ideal and *B* is an *N*-subgroup then (ii) is straightforward to show.

$(ii) \Rightarrow (i)$ Let *A* and *B* are two *N*-subgroups of *N* such that $AB \subseteq P$. Let *A* not be a subset of *P* and assume that $B^k \subseteq (P{:}A) = \{n \in N{:}\ An \subseteq P\} = S$. Since, *S* is an ideal of *N*, so if $r \in S$ and $n, n_1 \in N$, then for all $a \in A$, $a(-n + r + n) = -an + ar + an \in P$, as *P* is an ideal. Thus, $a[(n + r)\, n_1 - nn_1] = (an + ar)n_1 - ann_1 \in P$ which implies $Anr \subseteq Ar \subseteq P$. Hence $AS \subseteq P$ but we have assumed that $A \nsubseteq P$ implies $S \subseteq P$ so $B^k \subseteq S \subseteq P$.

Proposition 2.14 Let *P* be a 2-primary ideal and $A_1, \ldots, A_k$ are N-subgroups. Then $A_1 A_2 \ldots A_k \subseteq P$ implies $A_i^n \subseteq P$ for some $i \in \{1, \ldots, k\}$ and $n \in Z^+$.

Proof. Let $A_1 A_2 \ldots A_k \subseteq P$ and $A_1 \nsubseteq P$ such that $(A_2, \ldots, A_k)^n \subseteq (P{:}A_1)$. Thus $A_1 \cdot (P{:}A_1) \subseteq P$ implies $(P{:}A_1) \subseteq P$ given that *P* is 2-primary ideal. By using Proposition 2.13 (ii), we get $(A_2, \ldots, A_k)^n \subseteq P$. Similarly, we can repeat procedure for $A_2 \nsubseteq P$ and eventually $A_i^n \subseteq P$ for some $i \in \{1, \ldots, k\}$ where $i \neq 2$.

Definition 2.15 An ideal *P* of *N* is called a 3-primary ideal if for all $x, y \in N$, $xNy \subseteq P$ implies $x \in P$ or $y^n \in P$ for some $n \in Z^+$.

Example 2.16 Let *N* = {0, 1, 2, 3, 4, 5, 6, 7} be the (right) near-ring under the addition and multiplication defined in tables set 5.

Table 5. Set 5

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
1	2	3	0	1	7	6	4	5
2	2	3	0	1	5	4	7	6
3	3	0	1	2	6	7	5	4
4	4	7	5	6	2	0	1	3
5	5	6	4	7	0	2	3	1
6	6	4	7	5	1	3	0	2

·	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	0	2	2	2	0	0
3	0	3	2	1	5	4	6	7
4	0	4	2	5	4	5	6	7
5	0	6	2	4	5	4	6	7
6	0	6	0	6	0	0	0	0
7	0	7	0	7	2	2	0	0

Here, P = {0, 6} is a 3-primary ideal of N. One can check that for all $x, y \in N, xNy \subseteq P$ implies $x \in P$ or $y^n \in P$ for some $n \in Z^+$.

Definition 2.17 A proper ideal P of N is called (completely) c-primary ideal if for all $x, y \in N, xy \in P \Rightarrow x \in P$ or $y^n \in P$ for some $n \in Z^+$.

Example 2.18 Suppose N = {0, 1, 2, 3, 4, 5}, where " +" and " · " are defined in tables set 6.

Table 6. Set 6

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

·	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	5	1	0	5	1
2	0	4	2	0	4	2
3	0	3	3	0	3	3
4	0	2	4	0	2	4
5	0	1	5	0	1	5

Clearly, the right ideal P = {0, 2, 4} satisfies the conditions of c-primary ideal of N as $1 \cdot 3 = 0 \in P$ implies $3^2 = 0 \in P$ and $5 \cdot 3 = 0 \Rightarrow 3^2 = 0 \in P$.

Referring to Example 2.11, it is easy to verify that P = {0, 4} is a 0-primary ideal. Thus, every 1-primary ideal is a 0-primary ideal. From Example 2.11, we have observed that an ideal P = {0, 4} is 1-primary ideal but it is not a 1-prime ideal. Similarly, from Example 2.16, we can see that P = {0, 6} is 3-primary ideal but it is not a 3-prime ideal as $3N7 = \{0\} \subseteq P$ but 3 or 7 doesn't belong to P. Similarly, in Example 2.18, P = {0, 2, 4} is a c-primary ideal which is not a c-prime ideal. However, it is easy to verify that an ideal P is simultaneously c-primary, 3-primary, 2-primary, 1-primary and 0-primary ideal. Hence, we conclude that:

$$c - \text{primary} \Rightarrow 3 - \text{primary} \Rightarrow 2 - \text{primary} \Rightarrow 1 - \text{primary} \Rightarrow 0 - \text{primary}.$$

Referring to Example 2.11, {0, 4} is a υ-primary ideal (υ = 0, 1, 2, 3, c) which is the justification of above implications. But the converse doesn't hold true in the above implication. Because, in Example 2.2, {0} is 0-primary ideal but not 1-primary ideal. Similarly, {0} is not 3-primary ideal i.e, $3N4 = \{0\} \subseteq \{0\}$ but $3,\ 4 \notin \{0\}$ or $3^n,\ 4^n \notin \{0\}$ for some $n \in Z^+$. Further, one can check that {0} is not a c-primary ideal. In Example 2.11, {0} is a 1-primary ideal but not 2-primary. Similarly, {0} is not a 3-primary ideal, i.e, $3N7 = \{0\} \subseteq \{0\}$ but $3,\ 7 \notin \{0\}$ or $3^n,\ 7^n \notin \{0\}$.

After discussing different types of primary ideals in a near-ring, now we introduce υ-primary near-rings (υ = 0, 1, 2, 3, c).

Definition 2.19 A near-ring N is said to be a υ-primary near-ring (υ = 0, 1, 2, 3, c) if the ideal {0} is a υ-primary ideal of N.

We can state that N is a 0-primary near-ring, if for all ideals A and B of N, it holds that $AB \subseteq \{0\}$ implies $A \subseteq \{0\}$ or $B^n \subseteq \{0\}$. In a similar manner, one can define remaining υ-primary near-rings (υ = 0, 1, 2, 3, c).

Example 2.20 In Example 2.2, {0} is a 0-primary ideal of near-ring N. Consequently, N is a 0-primary near-ring.

Proposition 2.21 Each 0-prime near-ring is a 0-primary near-ring.

Example 2.22 Every integral near-ring is a prime near-ring and hence a primary (0-primary) near-ring.

Following (Pilz, 2011), if I is an ideal of N, then a prime radical of N is the intersection of all prime ideals containing I and is denoted by $\wp(I)$ i.e., $\wp(I) = \bigcap_{P \supseteq I} P$, where P is prime. Hence, if $n \in \wp(I) \Rightarrow \exists\ k \in \mathbb{N}: n^k \in I$. In other words, I is a semiprime ideal of N iff $\wp(I) = I$. Likewise for rings, we will see that if I is the υ-primary ideal (υ = 0, 1, 2, 3, c) of a near-ring, then its prime radical is the corresponding υ-prime ideal.

Example 2.23 Referring to Example 2.2, P = {0, 2} is 0-primary ideal and $\sqrt{\{0,2\}} = \{0, 1, 2, 3\}$ is a 0-prime ideal of N.

It is easy to verify that if an ideal I is a υ-primary, then its prime radical is a υ-prime which we have already seen in Example 2.23. But the converse doesn't hold true, i.e., if the prime radical of an ideal I is υ-prime then it is not necessary that I is a υ-primary ideal.

Proposition 2.24 Let I be both primary and semi-prime ideal of *N*. Then I is a prime ideal.

It is well known that an ideal I is a *c*-prime ideal (or completely prime), if $a, b \in N$, $ab \in I$ implies $a \in I$ or $b \in I$.

Definition 2.25 Let *Q* be a *c*-primary (completely primary) ideal of *N* such that $\sqrt{Q} = P$, where *P* is a *c*-prime ideal of *N*. Then we call *Q* a *cP*-primary ideal.

Definition 2.26 Let *Q* be a *cP*-primary ideal of *N*. Then, for $n \in N - Q$, it holds that $(Q: n) = \{a \in N: an \in Q\}$.

Proposition 2.27 Let *Q* be a *cP*-primary ideal of *N* and let $n \in N$. Then the relations below hold:

(i) If $n \in Q$, then $(Q: n) = N$;

(ii) If $n \notin Q$, then $(Q: n)$ is cP-primary ideal and $\sqrt{(Q: n)} = P$;

(iii) If $n \notin P$, then $(Q: n) = Q$.

Remark 2.28 Let *Q* be a *cP*-primary ideal of *N* such that $\sqrt{(Q: n)}$ is c-prime and $\sqrt{Q_i} = P_i$, then it must be contained in the set $\sqrt{(Q: n)}$ where $n \in N$.

We illustrate Proposition 2.27 and Remark 2.28 in the following example.

Example 2.29 Referring to Example 2.2, we have *Q* = {0, 2} is *c*-primary ideal. Then, only the possible *c*-prime ideal of *N* containing *Q* is the ideal *P* = {0, 1, 2, 3}and hence, it is a prime radical of *Q*. This implies *Q* is *cP*-primary. On the other hand, let $3 \in N$ and consider $(Q: 3) = \{n \in N: 3n \in Q\} = \{0, 1, 2, 3\}$, which is clearly a *c*-prime ideal. Hence $(Q: 3)$ is an associated *c*-prime ideal of a *c*-primary ideal *Q*. Thus, every associated *c*-prime ideal must be contained in $\sqrt{(Q: x)}$.

3. Applications of *υ*-Primary Ideals (*υ* = 0, 1, 2, 3, *c*) to Grap

In this section, we provide applications of υ-primary ideals (υ = 0, 1, 2, 3, *c*) to graph theory. In this regard, we provide characterizations of different graphs through υ-primary ideals (υ = 0, 1, 2, 3, *c*) of near-rings.

Definition 3.1 Let *P* be the υ-primary ideals (υ = 0, 1, 2, 3, *c*) of near-ring *N*. Then, the graph of υ-primary ideal is denoted by $G_{\upsilon-P}(N)$ and it consists of all the members of *N* as vertices and $x, y \in N$ are connected by directional edge from *x* to *y* (resp., *y* to *x*), if $xy \in P$ (resp., $yx \in P$). Similarly, if for any $x, y \in N$ such that $xy = yx \in P$, then there exists an undirected edge between *x* and *y*.

Example 3.2 Refer to Example 2.2, in which *P* = {0, 2} is 0-primary ideal of the near-ring *N*. Then the graph associated with 0-primary ideal denoted by $G_{0-P}(N)$ is shown in Figure 1.

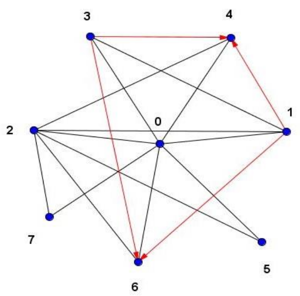

Figure 1. $G_{0-P}(N)$

According to the definition of 0-primary ideal $G_{I_1 I_2}(N) \sqsubseteq G_{0-P}(N)$ implies $G_{I_2^2}(N) \sqsubseteq G_{0-P}(N)$ but $G_{I_1}(N)$ is not a subgraph of $G_{0-P}(N)$. This shows the definition of 0-primary ideal by the graph which can be checked easily.

Example 3.3 In Example 2.11, P = {0, 4} is the 1-primary ideal of N. The graph of 1-primary ideal is denoted by $G_{1-P}(N)$ and is shown in Figure 2.

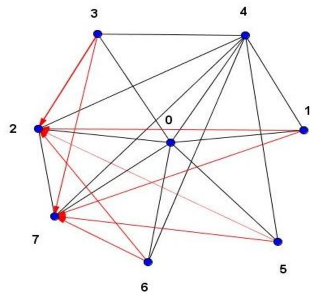

Figure 2. $G_{1-P}(N)$

In Example 2.11, the graph $G_{I_1 I_2}(N) \sqsubseteq G_{1-P}(N)$ implies that $G_{I_2^2}(N) \sqsubseteq G_{1-P}(N)$ and hence satisfies the definition of 1-primary ideal.

Example 3.4 In Example 2.16, P = {0, 6} is a 3-primary ideal. Then the graph $G_{3-P}(N)$ is presented in Figure 3.

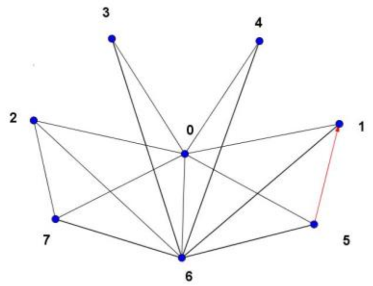

Figure 3. $G_{3-P}(N)$

The graph of $G_{3-P}(N)$ is different from the graph of 3-prime ideal because every prime ideal is primary but the converse does not hold in general.

Example 3.5 In Example 2.18, P = {0, 2, 4}is c-primary ideal. The graph of $G_{C-P}(N)$ is shown in Figure 4.

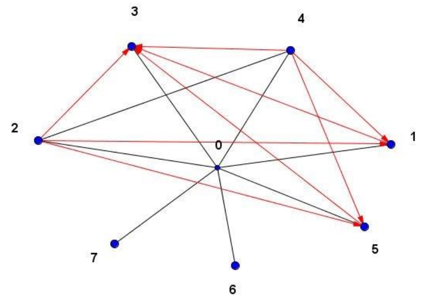

Figure 4. $G_{C-P}(N)$

Theorem 3.6 Let P be an ideal of near-ring. If P be 3-primary ideal, then P is a strong vertex cut of $G_{3-P}(N)$. If P is a 3-semiprimary ideal and P is strong vertex cut of $G_{3-P}(N)$, then P is 3-primary.

Proof. Let P be a 3-primary ideal of N. If $P = N$, then the proof is straightforward. Let us suppose that P is not equal to N and $a, b \in N - P$ with $a \neq b$. Let us suppose that there exists an edge between a and b of $G_{3-P}(N) \Rightarrow aNb \subseteq P$ or $bNa \subseteq P$. Since P is 3-primary ideal of N either $a \in P$ or $b^n \in P$, which contradicts our assumption that $a, b \in N - P$. So, as a result $G_{3-P}(N)$ has a strong vertex cut P. On the contrary, assume that P is 3-semiprimary ideal and a strong vertex cut of

$G_{3-P}(N)$. Then, we have to prove that P is 3-primary ideal of N. For this, let $a, b \in N$ such that $aNb \subseteq P$. As we know that P is 3-semiprimary ideal, $a = b \Rightarrow a \in P$. Let $a \neq b$ choose $a, b \in N - P$ for possible condition. As we know that P is strong vertex cut of $G_{3-P}(N)$, an edge does not exist between a and b of $G_{3-P}(N) \Rightarrow aNb \nsubseteq P$ and $bNa \nsubseteq P$. A contradiction arises, so $aNb \subseteq P$ implies $a \in P$ or $b^n \in P$. This proves that P is 3-primary ideal.

Lemma 3.7 (i) Let P be a 3-primary ideal of N and a be the vertex of $G_{3-P}(N)$. If deg$(a) =$ deg(0), then $a \in P$.

(ii) Suppose N is a zero-symmetric near-ring and vertex of $G_{3-P}(N)$ is a. If $a \in P$, then deg$(a) =$ deg(0).

(iii) Suppose N is a zero-symmetric near-ring and P be its 3-primary ideal. Then $a \in P$ if and only if deg$(a) =$ deg(0) in $G_{3-P}(N)$.

Proof. (i) $\Rightarrow$ Suppose deg$(a) =$ deg(0), then $aNb \subseteq P$ or $bNa \subseteq P$, for all $b \in N$ i.e., $a \neq b$. Assume that $aNb \subseteq P$, for all $b \in N$. If $P = N$, then $x \in P$. Let $P \neq N$ and choose $b \in N - P$. It is clear that P is 3-primary ideal of N and $aNb \subseteq P$ implies $a \in P$ or $b^n \in P$ for $n \in Z^+$, which proves condition (i). Now to prove (ii), let $a \in P$ and if $a = 0$, then the proof is straightforward. On the other hand, let $a \neq 0$ and deg$(a) \leq$ deg(0), then there is vertex b such that b is not adjacent to a in $G_{3-P}(N)$. Hence we can conclude that $aNb \nsubseteq P$ and $bNa \nsubseteq P$. Now according to our supposition $a \in P$ and P is an ideal of N implies $aN \subseteq P$. Thus $aNb \subseteq P$. As we know that N is zero-symmetric, then $Pb \subseteq P \Rightarrow aNb \subseteq P$ which contradicts our supposition. Hence deg$(a) =$ deg(0). Condition (iii) is follows form (i) and (ii).

Theorem 3.8 Suppose P is an ideal of N;

)i(*If N* is zero-symmetric and P is 3-primary ideal, then $G_{3-P}(N)$ is ideal symmetric.

)ii(If $G_{3-P}(N)$ is ideal symmetric with P is 3-semiprimary and for every $a \in N$, deg$(a) =$ deg(0) in $G_{3-P}(N) \Rightarrow a \in P$, then P is 3-primary and P is a strong vertex cut of $G_{3-P}(N)$.

Proof. To show condition (i), suppose a, b are any two vertices of $G_{3-P}(N)$ having an edge between a and b. Then, $aNb \subseteq P$ or $bNa \subseteq P$. Let $aNb \subseteq P$ and P is 3-primary ideal $\Rightarrow x \in P$ or $b^n \in P$, where n is any positive integer. Since N is a zero-symmetric, by using Lemma 3.7 (ii), deg$(a) =$ deg(0) or deg$(b) =$ deg(0). It implies $G_{3-P}(N)$ is an ideal symmetric. To verify (ii), suppose $a, b \in N$, $aNb \subseteq P$. According to given condition P is 3-semiprimary implies $a = b \Rightarrow a \in P$. Suppose $a \neq b$ then there exists an edge between a and b in $G_{3-P}(N)$. Since $G_{3-P}(N)$ is ideal symmetric, deg$(a) =$ deg(0) or deg(y)= deg(0) implies $a \in P$ or $b^n \in P$. As P is 3-primary ideal of N. So by Theorem 3.6, P is a strong vertex cut of $G_{3-P}(N)$.

4. Conclusions

In this paper, we have introduced the notions of υ-primary ideals (υ = 0, 1, 2, 3, c) in a near-ring, which generalize the υ-prime ideals (υ = 0, 1, 2, 3, c). We illustrate these defined ideals by examples and counter examples. We have investigated the relations of these ideals among each other as well. During this we have established that $c - \text{primary} \Rightarrow 3 - \text{primary} \Rightarrow 2 - \text{primary} \Rightarrow 1 - \text{primary} \Rightarrow 0 - \text{primary}$, but the converse implications are not true. Furthermore, we have proved some logical results and verified them through examples. Finally, we have studied different types of graphs associated with υ-primary ideals (υ = 0, 1, 2, 3, c) and also proved some algebraic results by using the concepts of graph theory. In the future, υ-primary ideals (υ = 0, 1, 2, 3, c) and their defined relations with other ideals in near-ring will help to differentiate and introduce more algebraic structure, which process has not yet been initiated in near-rings. One can also study the sequential machine via using these newly established ideals and the graphs associated with them.

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