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Original Article

Study of pioneer anomaly for anisotropic fluid in off-diagonal metric

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Abstract

We obtain the solutions of Einstein's field equations for an anisotropic fluid in a static spherically symmetric metric with off-diagonal elements. We also study the pioneer anomaly with the help of the obtained solutions. First, we derive analytical solutions to study sudden change in gravity in the radial and tangential directions of a non-homogeneous gravitational system by considering Π as a constant. Then we find other solutions by considering Π as a function of r in the space of gravitating object. We extend our solutions in the numerical limits to verify the unusual behavior of gravity at larger distances for the inhomogeneous system. We show a sudden increase in radial pressure for $r\epsilon[0,20]$ as well as the tangential pressure for $\theta\epsilon[0,\pi]$, which may be treated as a factor of pioneer anomaly. We obtain the anomalous acceleration $2.493\times10^{-11}m/s^2$ at a distance of 20 AU from the Sun, which has better accuracy than other theoretical calculations when it is compared to the observed value. We also derive an expression for cosmological constant in terms of r. We establish that the cosmological constant is a function of r, which seems to be variable but its variability diminishes for large values of r. The results of this study are interesting and quite new. The selected spherically symmetric metric with off-diagonal elements provides a modified method to observe the phenomena of general relativity.

Keywords: anisotropic, spherically symmetric metric, off-diagonal elements, pioneer anomaly, cosmological constant

1. Introduction

In 1915 Einstein proposed the general theory of relativity. Einstein expressed that the gravity is a result of the distortion in spacetime caused by massive objects (Einstein, 1915a). Later, Einstein presented the field equations to relate the geometry of spacetime with the distribution of matter inside it (Einstein, 1915b). Einstein explained that the gravity is more than just a force contradicting Newton's concept of gravity. He determined the gravity as the distortion of spacetime due to the presence of matter or energy, which can be described using a metric tensor. The Newtonian gravitation is a special case of the metric tensor in Einstein's theory.

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The metric tensor with diagonal elements only is well defined and achieves the Newtonian gravitation. Newtonian gravitation presumes homogeneous gravity for the entire universe. Therefore, Newtonian gravitation is insufficient for describing phenomena in which the gravity is inhomogeneous, e.g. the phenomenon of unusual high-speed arms of spiral galaxy or the pioneer anomaly in which a sudden increase in gravity at a larger distance is observed; these cannot be explained by using the gravity tensor with diagonal elements only (Murad, 2011). A spiral galaxy is a flattened rotating galaxy with pinwheel-like arms of interstellar material and young stars, winding out from its central bulge. M100, M101 (Kuntz, 2003), NGC 1232, NGC 1365, NGC 2903 (Bresolin, 2005) etc. are examples of spiral arm galaxies.

The pioneer anomaly is the unusual behaviour of acceleration of Pioneer 10 and Pioneer 11 spacecrafts after travelling long distances in the space, which was observed in 1980 and investigated in 1994 (Nieto & Anderson, 2005; Nieto & Turyshev, 2004; Turyshev *et al*., 2006; Turyshev, Nieto, &

Anderson, 2005). The pioneer anomaly was deceleration of the Pioneer spacecrafts (Varieschi, 2012), which in fact is in violation of Newton's inverse square law (Turyshev, 2010). Initially it was believed that the cause of pioneer anomaly was radiative anisotropy (Rievers & Lammerzahl, 2011). It was also discussed that pioneer anomaly is a gravitational phenomenon (Iorio, 2010) and the metric, which is not Minkowski flat, such as the FLRW metric, can explain the pioneer anomaly in a straightforward manner (Iorio, 2010). However, advanced studies undermine the role of gravitation in pioneer anomaly (Fienga *et al*., 2009; Iorio, 2007; Iorio & Giudice, 2006; Standish, 2008; Tangen, 2007) but others did not rule out the role of gravity in the pioneer anomaly (AlMosallami, 2012; Anderson & Nieto, 2009; Anderson *et al*., 2002; Iorio, 2015; Jaekel & Reynaud, 2008; Page, Wallin & Dixon, 2009; Siutsou & Tomilchik, 2009). We selected a spherically symmetric off-diagonal metric for studying the gravitational cause of pioneer anomaly.

It is not only the pioneer anomaly but also there are many other geophysics situations where a metric tensor with off-diagonal elements is needed (Jefimenko, 2006; Murad, Lavrentiev, Dyatlov, Fadeev, & Kostova, 2000). With sufficient evidence of the need for off-diagonal metrics, studies have attempted to define the off-diagonal metric tensor and tried to prove its validity.

Hobson, Efstethiou, and Lasenby (2007) proposed a geodesic equation in spherically symmetric metric tensor with off-diagonal elements as follows.

$$\mathrm{ds}^2 = \mathrm{g}_{00}(\mathrm{r})\mathrm{c}^2\mathrm{dt}^2\text{-}\mathrm{g}_{11}(\mathrm{r})\mathrm{dr}^2\text{-}2\mathrm{cg}_{01}(\mathrm{r})\mathrm{dtdr}\text{-}\mathrm{l}(\mathrm{r})\mathrm{r}^2(\mathrm{d}\theta^2 + \mathrm{d}\phi^2\sin^2\theta). \tag{1}$$

Later, Friedmann and Stave (2019) modified equation (1) and suggested a new geodesic equation with off-diagonal metric of the following form

$$\mathrm{ds}^2 = (1\text{-u})\mathrm{dt}^2\text{-}2\mathrm{g}_{01}\mathrm{dtdr}\text{-}\frac{(\mathrm{g}_{01}^2\text{-}1)}{1\text{-u}}\mathrm{dr}^2\text{-}\mathrm{r}^2\mathrm{d}\theta^2\text{-}\mathrm{r}^2\sin^2\theta\mathrm{d}\phi^2. \tag{2}$$

They also showed that their metric passed the classical tests of general relativity. With the above studies available to us, we must consider a spherically symmetric metric with off-diagonal elements in case of inconsistently varying gravity due to any gravitating object.

Moreover, in our universe, we notice the atmospheric density and hence atmospheric pressure within the perihelion and aphelion of a gravitating object possessing orbital motion. Studies have shown variations in density, pressure and hence in gravitation in different directions of gravitating objects as well as in the atmosphere (Deng, Qidley, & Wang, 2008). The shifting of the Mercury perihelion forward (Einstein, 1915b; Price & Rush, 1979) and expansion of the universe (Bahcall, 2015; Dil, 2016; Kirshner, 2021) show the varying nature of density, pressure, and gravity of the gravitating objects. These theories guided us to study an anisotropic fluid with off-diagonal metric.

The study of non-homogeneous gravity is an interesting topic in the general theory of relativity and cosmology, in the contemporary field of astrophysics. The mathematical physics researchers (Maurya, Maharaj, Kumar & Prasad, 2019) have shown a lot of interest in this field. Recent advances (Maurya, Banerjee & Hansraj, 2018) in this topic motivated us also to study the inhomogeneous gravitational system.

In this work we took our defined static spherically symmetric metric with off-diagonal elements for anisotropic fluid. In the first part, we derive analytical solutions of Einstein field equations in the off-diagonal metric defined by us. In the second part, we calculate numerical solutions by limiting the parameter values. In the third part we discuss the pioneer anomaly and calculate the anomalous acceleration using the results obtained in the second part. In the fourth part we discuss the cosmological constant.

2. Anisotropic Solution

The Einstein field equations with cosmological constant can be given by (Blau, 2011)

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = kT_{\mu\nu}, \tag{3}$$

where, $R_{\mu\nu}$ is Ricci tensor, $g_{\mu\nu}$ is metric tensor, Λ is cosmological constant, k is a constant equal to $\frac{8\pi G}{c^4}$ with G the well-known gravitational constant, c is speed of light, and $T_{\mu\nu}$ is energy-momentum tensor.

Later, it was found that the cosmological constant was not significant and it was set to zero. Finally, the Einstein field equations in four dimensions can be written as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = kT_{\mu\nu}, \tag{4}$$

where, $G_{\mu\nu}$is Einstein tensor.

Now, we take our off-diagonal metric for a symmetric sphere of anisotropic fluid as

$$\mathrm{ds}^2 = \mathrm{Udt}^2 - \mathrm{Wdtdr} - \frac{4-\mathrm{W}^2}{4\mathrm{U}}\mathrm{dr}^2 - \mathrm{r}^2\mathrm{d}\theta^2 - \mathrm{r}^2\sin^2\theta\mathrm{d}\phi^2, \tag{5}$$

with assumptions similar to Friedman and Stav (2019) and additionally setting $c = 1$. U and W are functions of r only. The determinant of time-radial part of the metric given in equation (5) is -1. The geodesic equation in such metric exactly represents the precision of a planetary orbit, perihelion shift in a double star system, deflection of light and Shapiro time delay (Friedman & Stav, 2019).

We are interested in finding the spherical spacetime for a fluid of anisotropic nature. We achieve this by solving the Einstein field equations where the energy-momentum tensor is non-zero. In order to proceed and obtain solutions, we need some basic components and tensor used in general relativity. Here, we are taking Ricci tensor as $R_{\mu\nu}$, Ricci scalar as R, Einstein tensor as $G_{\mu\nu}$ and energy-momentum tensor as $T_{\mu\nu}$, for our defined spherically symmetric metric with off-diagonal elements.

We calculate the Ricci scalar as follows

$$R = U'' + \frac{4}{r}U' + \frac{2}{r^2}U - \frac{2}{r^2}. \tag{6}$$

Ricci tensor components can be given as follows

$$R_{tt} = \frac{1}{2}UU'' + \frac{U}{r}U', \tag{7}$$

$$R_{tr} = -\frac{1}{4}WU'' - \frac{W}{2r}U' = R_{rt}, \tag{8}$$

$$R_{rr} = \frac{W^2}{8U}U'' - \frac{1}{2U}U'' + \frac{W^2}{4rU} - \frac{1}{4rU}U', \tag{9}$$

$$R_{\theta\theta} = -rU' - U + 1, \tag{10}$$

$$R_{\phi\phi} = sin^2\theta R_{\theta\theta}. \tag{11}$$

The non-zero Einstein tensor components are

$$G_{tt} = \frac{U}{r^2}(-rU' - U + 1), \tag{12}$$

$$G_{tr} = \frac{W}{2r^2}(rU' + U - 1) = G_{rt}, \tag{13}$$

$$G_{rr} = \frac{(W^2-4)}{4r^2U}(-rU' - U + 1), \tag{14}$$

$$G_{\theta\theta} = r\left(\frac{1}{2}rU'' + U'\right), \tag{15}$$

$$G_{\phi\phi} = G_{\theta\theta}sin^2\theta. \tag{16}$$

Now, we are taking an inhomogeneous fluid with the following assumptions for the components of energy-momentum tensor $T_t^t = \rho$; $T_r^t = -P_{tr} = T_t^r$; $T_r^r = -P_r$; $T_\theta^\theta = -P_\theta$; $T_\phi^\phi = -P_\phi$ and consequently, $T_{tt} = \rho U$; $T_{tr} = \frac{W}{2}P_{tr} = T_{rt}$; $T_{rr} = \frac{(4-W^2)}{4U}P_r$; $T_{\theta\theta} = r^2P_\theta$; $T_{\phi\phi} = P_\phi r^2 sin^2\theta$, where, $\rho, P_r, P_\theta, P_\phi$, and P_{rt} are proper energy density, radial pressure, tangential pressure, co-tangential pressure, and pressure in time varying radial direction respectively.

With the assumptions for energy-momentum tensor made above for a locally anisotropic fluid and using equations (12) to (16), we obtain the following field equations

$$\frac{1}{r^2}\left(-rU' - U + 1\right) = 8\pi G\rho, \tag{17}$$

$$\frac{1}{r^2}\left(rU' + U - 1\right) = 8\pi GP_{tr} = 8\pi GP_{rt}, \tag{18}$$

$$\frac{1}{r^2}\left(rU' + U - 1\right) = 8\pi GP_r, \tag{19}$$

$$\frac{1}{r}\left(\frac{1}{2}rU'' + U'\right) = 8\pi GP_\theta. \tag{20}$$

Now, by simply following the Herrera algorithm (Herrera, 2008), we subtract equation (20) from equation (19) to get

$$\Pi = -\frac{1}{2}U'' + \frac{1}{r^2}U - \frac{1}{r^2}, \tag{21}$$

where, $\Pi = 8\pi G(P_r - P_\theta)$.

2.1 Solution for static Π

If we consider a local coordinate system then Π is constant in the region of fixed coordinates (t, r, θ, ϕ). In this case, equation (21) is a simple differential equation of order two and can be solved easily. Solving this equation gives the following result

$$U = 1 + \frac{C_1}{r} + r^2\left(\frac{2\Pi}{9} + C_2\right) - \frac{2}{3}r^2\Pi logr. \quad (22)$$

This solution is dependent upon r and is consistent when $\Pi = 0$, and C_1 and C_2 are integration constants. Therefore,

$$U = 1 + \frac{C_1}{r} + C_2 r^2$$

First, we take $W = 0$ and the solution obtained in equation (22) to write our metric given by equation (5), and the metric takes the following form

$$ds^2 = \left(1 + \frac{C_1}{r} + C_2 r^2\right)dt^2 - \frac{1}{\left(1 + \frac{C_1}{r} + C_2 r^2\right)}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2. \quad (23)$$

If we take $C_1 = -2M$ and $C_2 = -\frac{1}{3}\Lambda$, where M is the mass of source and Λ is cosmological constant, this solution is exactly similar to the solution given earlier in the metric of the Schwarzschild-de Sitter spacetime (Kagramanova, 2006; Kerr, 2003; Sereno, 2006) and static solution of Einstein's field equations (Zubairi, Romero & Weber, 2015).

Similarly, for a constant value of W that is $W = K$, the solution metric becomes

$$ds^2 = \left(1 + \frac{C_1}{r} + C_2 r^2\right)dt^2 - Kdtdr - \frac{(4 - K^2)}{\left(1 + \frac{C_1}{r} + C_2 r^2\right)}dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2. \quad (24)$$

We took W to be either zero or a constant because W does not appear in any of the field equations.

Further, it is manifestly clear from the field equations that $P_{tr} = P_{rt} = P_r$. Therefore, we are solving equation (17) only taking the energy density as a constant. We get the solution,

$$U = 1 + \frac{C_3}{r} - \frac{8\pi G\rho}{3}r^2. \quad (25)$$

Putting $C_3 = 2MG$ in equation (25) and doing a little calculations, we get

$$U = 1 - \frac{1.99GM}{r}. \quad (26)$$

Further from the local conservation law of energy-momentum (Stephani, Kramer, Maccallum, Hoenselaers, & Herlt, 2003), $T^{\mu}_{\nu;\mu} = 0$. We also know that

$$T^{\mu}_{\nu;\mu} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\mu}}\left(\sqrt{-g}T^{\mu}_{\nu}\right) - \Gamma^{\alpha}_{\nu\mu}T^{\mu}_{\alpha}.$$

Using the above condition for conservation and the formula of energy-momentum tensor, we obtain the following four equations

$$P'_{tr} + \frac{1}{4}WU'(P_r - \rho) + \left(\frac{1}{2U}U' - \frac{1}{2}UU' - \frac{W^2}{8U}U' + \frac{2}{r}\right)P_{tr} = 0 \quad (27)$$

$$P'_r + \left(\frac{1}{2U}U' - \frac{W^2}{8U}U'\right)(\rho + P_r) + \left(\frac{1}{4}WU' + \frac{w}{4U^2}U' - \frac{W^3}{16U^2}U' + \frac{1}{2U}W'\right)P_{tr} + \frac{1}{r}\left(2P_r - P_\theta - P_\phi\right) = 0 \quad (28)$$

$$P'_\theta + cot\theta\left(P_\theta - P_\phi\right) = 0, \quad (29)$$

$$P'_\phi = 0. \quad (30)$$

Equation (30) produces the result $P_\phi = C$, where C is an integration constant.

Now, taking equation (29), we get,

$$P'_\theta + P_\theta cot\theta = Ccot\theta. \quad (31)$$

This is a simple linear equation of order one which can be solved easily. The result is

$$P_\theta = C + \frac{C_4}{sin\theta}. \quad (32)$$

Now taking $W = 0$ and $P_{tr} = P_r$ as discussed above and putting these in equation (27), we get

$$P'_r + \left(\frac{1}{2U}U' - \frac{1}{2}UU' + \frac{2}{r}\right)P_r = 0, \quad (33)$$

Putting the value of U from equation (26) in equation (33), we get

$$P'_r + \left(\frac{1}{2}\frac{\frac{1.99GM}{r^2}}{1 - \frac{1.99GM}{r}} - \frac{1.99GM}{r^2}\left(1 - \frac{1.99GM}{r}\right) + \frac{2}{r}\right)P_r = 0,$$

this equation produces the following result

$$P_r = \frac{e^{0,99\left(\frac{G^2M^2}{r^2} - \frac{GM}{r}\right)}}{r^{3/2}\sqrt{r - 1.99GM}}C_5, \tag{34}$$

where, C_5 is an integration constant. Upon setting $C_5 = 1$ in equation (34), we get

$$P_r = \frac{e^{0,99\left(\frac{G^2M^2}{r^2} - \frac{GM}{r}\right)}}{r^{3/2}\sqrt{r - 1.99GM}} \tag{35}$$

Equation (35) is the final equation of state of the radial pressure and mass energy-density.

2.2 Solution for non-static Π

Now we take the region of $(t + dt,\ r + dr,\ \theta + d\theta,\ \phi + d\phi)$ for all the possible values of $(\mathrm{d}t, \mathrm{d}r, \mathrm{d}\theta, \mathrm{d}\phi)$ in the space of the gravitating object, and then equation (21) can be solved with a little difficulty as follows

$$U = \frac{C_6}{r} + C_7r^2 + \frac{2}{3r}\int_1^r \left(1 + x^2\Pi(x)\right)dx - \frac{2r^2}{3}\int_1^r \frac{1 + y^2\Pi(y)}{y^3}dy \tag{36}$$

Therefore, from the equations (5) and (36), we obtain

$$\begin{aligned} ds^2 = &\left(\frac{C_6}{r} + C_7r^2 + \frac{2}{3r}\int_1^r \left(1 + x^2\Pi(x)\right)dx - \frac{2r^2}{3}\int_1^r \frac{1 + y^2\Pi(y)}{y^3}dy\right)dt^2 \\ &- \frac{1}{\frac{C_6}{r} + C_7r^2 + \frac{2}{3r}\int_1^r \left(1 + x^2\Pi(x)\right)dx - \frac{2r^2}{3}\int_1^r \frac{1 + y^2\Pi(y)}{y^3}dy}dr^2 \\ &- r^2(d\theta^2 + sin^2\theta d\phi^2) \end{aligned} \tag{37}$$

In this way, from equation (37), any solution for non-static Π in case of an anisotropic distribution of fluids in a gravitating object can be completely determined by the estimation of one generating function Π.

Now for any perfect fluid system, we can take $\Pi = 0$, hence equation (36) turns to

$$U = 1 + \frac{C_6}{r} + C_7r^2 \tag{38}$$

Thus, from equation (38) and equation (5), we get

$$ds^2 = \left(1 + \frac{C_6}{r} + C_7r^2\right)dt^2 - \frac{1}{1 + \frac{C_6}{r} + C_7r^2}dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{39}$$

From the equations (17), (19) and (38) it is clear that $\rho > 0$ and $\rho > P_r$, $P_{tr,}P_\theta$ are monotonically decreasing functions of r, which formulates the stability of the solution with meaningful physical applications.

3. Pioneer Anomaly and General Relativity

We have obtained the solution of $U = U(r)$ in equation (22) as a function of r. Therefore, we may consider U as function of r only. If, with the initial condition, taking $C_1 = \frac{2MG}{c^2}$ and $C_2 = \frac{2MG}{c^2}$ (Narlikar, 2002) and approximating $\Pi = 0$ then for $r\epsilon[0, 10]$ we observe a sudden change in U. The graph of this function is shown in Figure 1.

Now, to obtain a numerical approximation of the radial pressure, putting $C_5 = 1,\ GM = 1$ and taking r in a range $[0, 20]$ in equation (34), we observe a continuous drop till $r = 20$ in the radial pressure after a sudden drop at approximately $r = 5$. This situation well explains the Pioneer anomaly studied earlier (Philip, 1997), which is evidently clear from the following figure.

Similarly, from equation (32), we notice a sudden change in tangential pressure at the extremities of tangential angle in the range $[0, \pi]$. We see the following graph of tangential pressure in the specified range.

3.1 Anomalous acceleration

The constant acceleration acting on the spacecraft has been measured as $8.5 \times 10^{-8} m/s^2$ (Anderson *et al*., 1998). There is one another interpretation in which it is believed that there is a constant acceleration of $8.74 \pm 1.33 \times 10^{-10} m/s^2$ directed towards the Sun (Anderson *et al*., 2002).

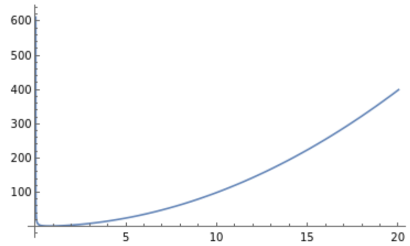

Figure 1. Plot of $U(r)$ when $\Pi = 0$, $r\epsilon[0, 20]$.

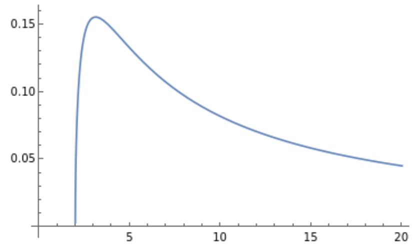

Figure 2. Plot of P_r when $C_5 = 1, GM = 1$ and $r\epsilon\ [0, 20]$.

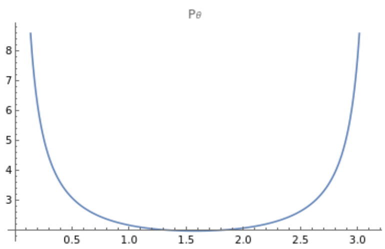

Figure 3. Plot of P_θ, when $C = 1$ and $\theta\epsilon[0, \pi]$.

Many researchers have obtained clock acceleration in different ways (Feldman, 2013; Ranada, 2004). We are using our off-diagonal spherically symmetric metric for a perfect fluid,

$$c^2 d\tau^2 = ds^2, \tag{40}$$

where τ is the proper time.

Taking equations (39) and (40) and considering that the solid angle remains constant at larger distances, we obtain

$$d\tau^2 = Ydt^2 - \frac{1}{Yc^2}dr^2, \tag{41}$$

where

$$Y = 1 + \frac{C_1}{r} + C_2 r^2,$$

by taking $C_1 = C_2$, we get

$$Y = 1 + \frac{2GM}{rc^2} + \frac{2GM}{c^2}r^2, \tag{42}$$

For a very large value of *r* equation (41) can further be approximated as

$$\mathrm{d}\tau = \left(1 + \frac{3}{2}\left(\frac{dr}{cdt}\right)^2\right)\left(\frac{1}{r} + r^2\right)\frac{GM}{c^2}dt. \tag{43}$$

In terms of an effective speed of light *c*(*t*) in coordinate time *t*, the clock rate difference can be given as,

$$\mathrm{c(t)} = \mathrm{c}\left(1 + \frac{3}{2}\left(\frac{dr}{cdt}\right)^2\right)\left(\frac{1}{r} + r^2\right)\frac{GM}{c^2}. \tag{44}$$

Now, dividing by λ, the wavelength, we get,

$$\mu_o = \mu_s\left(1 + \frac{3}{2}\left(\frac{dr}{cdt}\right)^2\right)\left(\frac{1}{r} + r^2\right)\frac{GM}{c^2}, \tag{45}$$

where μ_o is observed frequency and μ_s is source frequency.

In case of pioneer the one-way Doppler formula for a source moving away from the observer with velocity *v* can be given as

$$\mu_o = \mu_s\left(1 - \frac{v}{c}\right).$$

Hence, by ignoring $\frac{1}{c}\frac{dr}{dt}$ the Doppler formula becomes

$$\mu_o = \mu_s\left(\frac{1}{r} + r^2\right)\frac{GM}{c^2}\left(1 - \frac{v}{c}\right). \tag{46}$$

Now, using the Wong (2019) methodology in Newtonian approximation without the cosmological constant Λ, to calculate acceleration a_p we obtain

$$a_p = 2GM\left(\frac{1}{r^2}\right)\frac{\dot{r}}{c}, \tag{47}$$

where $\dot{r}$ is the velocity of source moving away from the observer. The value of $\frac{\dot{r}}{c}$ is of the order of 10^{-4}.

We obtain $a_p = 2.493 \times 10^{-11} m/s^2$ at a distance of 20 AU. This is a better calculated value than the others. The previously calculated a_p by a few researchers are summarized in Table 1.

4. The Cosmological Constant

If we take $C_2 = -\frac{1}{3}\Lambda$ as we have discussed in connection with equation (23), we get

$$\Lambda = \frac{8\pi G\rho}{3}, \tag{48}$$

where,

$$\rho = \frac{M}{\frac{4}{3}\pi r^3} \tag{49}$$

Using equations (48) and (49), we can write the cosmological constant as follows

$$\Lambda = \frac{2GM}{r^3}. \tag{50}$$

Equation (50) shows that the cosmological constant taken by Kagramanova, Kunz, and Lammerzahl (2006) is dependent on r. In fact, the cosmological constant has turned into a function of the variable radial coordinate and produces a varying effect with variation in r, but slowly it gains a nearly constant nature or an infinitesimally small increasing nature at large values of r. The value of cosmological constant at the large 100 AU distance is of the order of 10^{-28}. The nature of Λ in a smaller range and in a larger range is shown in Figure 4 and Figure 5 respectively.

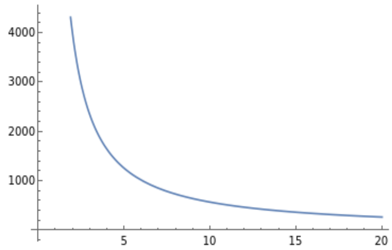

Figure 4. Plot of a_p for $0 \le r \le 20$.

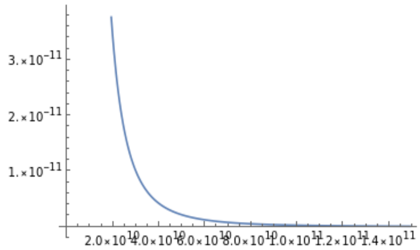

Figure 5. Plot of Λ, when $r\epsilon[0, 1AU]$.

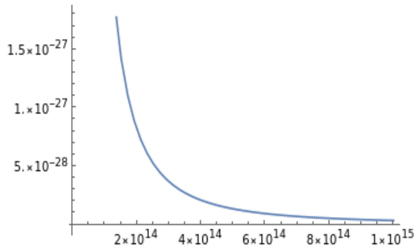

Figure 6. Plot of Λ, when $r\epsilon[0, 100AU]$.

Table 1. Numerical values of a_p calculated with different metrics

Author	Distance (inAU)	a_p
AlMosallami (2012)	25.2	$2.62\times 10^{-10} m/s^2$
Wong (2019)	30	$3.75\times 10^{-10} m/s^2$
Ferreira (2013)	20	$2.5\times 10^{-10} m/s^2$

All the graphically shown approximations described above can also be tabulated as follows. In Table 2, we are showing the approximate values of radial pressure, the component *U* of the metric tensor and the cosmological constant in the range $10 \le r \le 100$.

Table 2. Numerical approximations of $\Pi(r)$, U(r), and $\Lambda(r)$ for $10 \le r \le 100$]

r (in AU)	P(r)	$\Pi(r)$	U(r)	$\Lambda(r)$
10	1.00428×10^{-9}	1.04953×10^{-26}	8.90378×10^{37}	6.21327×10^{-25}
20	3.53945×10^{-10}	2.55854×10^{-27}	3.56151×10^{38}	1.55014×10^{-25}
30	1.92549×10^{-10}	1.12814×10^{-27}	8.01342×10^{38}	6.92628×10^{-26}
40	1.25038×10^{-10}	6.32116×10^{-28}	1.42464×10^{39}	3.91742×10^{-26}
50	8.94608×10^{-11}	4.03623×10^{-28}	2.22594×10^{39}	2.51939×10^{-26}
60	6.80514×10^{-11}	2.79867×10^{-28}	3.20536×10^{39}	1.75712×10^{-26}
70	5.40010×10^{-11}	2.05394×10^{-28}	4.36285×10^{39}	1.29588×10^{-26}
80	4.41981×10^{-11}	1.57127×10^{-28}	5.69842×10^{39}	9.95548×10^{-27}
90	3.70398×10^{-11}	1.24072×10^{-28}	7.21200×10^{39}	7.89036×10^{-27}
100	3.16247×10^{-11}	1.00448×10^{-28}	8.90378×10^{39}	6.40916×10^{-27}

5. Conclusions

We took an off-diagonal spherically symmetric metric to obtain solutions of Einstein field equations in order to study inhomogeneously gravitating objects. In this metric, we obtained a solution $1+\frac{C_1}{r}+C_2r^2$, which is analogous to the solution $1-\frac{2M}{r}-\frac{1}{3}\Lambda r^2$ given by Kagramanova *et al*. (2006) to study the pioneer anomaly. By this comparison we were able to show that the cosmological constant is of varying nature and dependent on *r*, and for large values of *r* the cosmological constant is growing very slowly or is nearly constant. In a smaller range of r namely $[0, 1AU]$ and taking all the integration constants as 1, the cosmological constant is decreasing in the order of 10^{-11} . But in a larger range, as we observe in Figure 5, the cosmological constant shows extremely slow variation after $r = 30$ AU. This observation reveals that the cosmological constant is not achievable in the form of a perfect invariant. Hence, the cosmological constant has been turned into an expansion variable. Therefore, we can infer that the cosmological constant in the form obtained by us has some impact in the pioneer anomaly.

We also observed a sudden change in radial pressure in the range $r\epsilon[0, 20]$ and an increase in tangential pressure at the extremities of the tangential angle in the range $0 \le \theta \le \pi$. The radial pressure decreases from $r = 10$ AU to $r = 100$ AU in the range from 1.00428×10^{-9} to 3.16247×10^{-11}. Similarly, the tangential pressure exhibits a sudden change at $\theta = \pi$. The metric component $U(r)$ exhibits a consistently increasing behavior from $U(10)$ to $U(100)$ with *r*. All these results establish the decline in gravity due to a gravitating object in any arbitrary range, which is a cause of the pioneer anomaly. We estimated the value of a_p as $2.493 \times 10^{-11} m/s^2$ at 20 AU from the Sun, which is approximately equal to the observed data during 7.8 years of the study of pioneer anomaly.

We also obtained another solution $U = \frac{C_1}{r} + C_2r^2 + \frac{2}{3r}\int_1^r\left(1 + x^2\Pi(x)\right)dx - \frac{2r^2}{3}\int_1^r \frac{1+y^2\Pi(y)}{y^3}dy$ with one generating function Π, which has physical significance in some certain conditions of energy density and pressure. If we restrict the object in an isotropic system then we got the solution $U = 1 + \frac{C_1}{r} + C_2r^2$, which is a modified form of the solutions (Gabbanelli, Rincón, & Rubio, 2018; Kramer, Stephani, MacCallum., & Herlt1980; Singh, Bhar, & Pant, 2016) given for anisotropic fluid.

Moreover, from Table 1, we conclude that an off-diagonal metric is better suited for the study of behavior of an object in motion under an inhomogeneous gravitating body.

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