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Original Article

Weighted hesitant bipolar-valued fuzzy soft set in decision-making

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Abstract

In this research work, we have introduced a new notion of weighted hesitant bipolar-valued fuzzy soft set (WHBFSS) as a generalization of hesitant bipolar-valued fuzzy soft set (HBFSS), and we examine some of its fundamental properties in detail. We've also defined some novel notions of the root mean square difference operator (RMSDO), root mean square difference score matrix (RMSDSM), and weighted score, and using these novel notations, we have proposed an advanced and adjustable decision-making method (DMM) for solving real-life decision-making problems (DMPs) based on both HBFSS and WHBFSS. A real-life example is provided to demonstrate the validity of our suggested method. Finally, a comparative analysis of our approach with an existing method is provided.

Keywords: decision-making, soft set, fuzzy set, hesitant fuzzy set, bipolar fuzzy set

1. Introduction

Molodtsov (1999) developed the basic results of soft sets (SSs) and successfully applied them to a variety of fields, including the smoothness of functions, operations analysis, game theory, Riemann integration, probability, and so on. To solve DMPs, Maji, Biswas, and Roy (2002) used SSs for the first time. Recently, several authors have looked into properties and applications of SSs more broadly. Alcantud and Santos-García (2017) presented a new criterion for SSs-based DMPs under incomplete information, and Dalkılıç (2021) proposed a novel approach to SSs in DMPs under uncertainty.

The idea of the fuzzy set (FS) was started by Zadeh (1965), and thereafter many new approaches and ideas have been offered to deal with imprecision and ambiguity, such as hesitant fuzzy set (HFS), intuitionistic fuzzy set (IFS) (Atanassov, 1986), bipolar-valued fuzzy set (BFS) (Lee, 2000), and so on. Torra (2010) first introduced the theory of HFSs, and later on, Rodryguez, Martynez, Torra, Xu, and Herrera (2014) presented the state of the art and future directions of HFSs. Xia and Xu (2017) proposed hesitant fuzzy (HF) information aggregation in DMPs and also studied some properties of HFSs. Ren and Wei (2017) proposed an MCDM algorithm with a prioritization relationship and dual HF-decision information. Xu and Zhou (2017) presented consensus building with a group of decision-makers (DMs) in a hesitant probabilistic fuzzy environment. Liu and Zhang (2017, 2017a) suggested an extended multi criteria decision-making (MCDM) technique using neutrosophic HF-information and also proposed another MCDM technique using neutrosophic HF-heronian mean aggregation functions. Alcantud and Torra (2018) presented some decomposition theorems as well as extension principles for HFSs. Naz and Akram (2019) suggested a novel DMM based on HFSs and graph theory. Alcantud and Giarlotta (2019) studied the necessary and possible HFSs as well as proposed a novel model for group DMPs.

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Maji, Biswas, and Roy (2001) described fuzzy soft set (FSS), which is a hybrid of FS and SS. The applications of FSSs have been gradually concentrated by using these concepts. Feng, Jun, Liu, and Li (2010) introduced an adjustable DMM to solve FSS-based DMPs. Wang, Li, and Chen (2014) defined the hesitant fuzzy soft set (HFSS) and proposed its applications in MCDM. The topic of intertemporal-FSS selection was first raised by Alcantud and Muoz Torrecillas (2017). Peng and Dai (2017) suggested some HF-soft DMMs using COPRAS, MABAC, and WASPAS. Using revised aggregation functions, Peng and Li (2019) suggested a DMM for HF soft DMPs. Gao and Wu (2021) defined filters and their applications in topological spaces formed by FSSs. Dalkiliç (2021a) defined topology on virtual fuzzy parameterized-FSSs. Bhardwaj and Sharma (2021) described an advanced uncertainty measure using FSSs and suggested an application in DMPs. Fatimah and Alcantud (2021) presented the idea of multi-fuzzy N-SS and its applications in DMPs. Later on, Das and Granados (2022) introduced a new theory on FP-IFS multisets and suggested an adjustable approach based on FP-IFS multisets; also, Das, Granados and Bhattacharya (2022) defined some new operations on FSSs and studied their applications in decision-making. Recently, Das and Granados (2023) introduced a new notion of IFP-intuitionistic multi fuzzy N-soft set and induced IFP-hesitant N-soft set and also studied their applications to solve real-world DMPs. Granados, Das and Osu (2023) defined weighted neutrosophic soft multisets and studied their application to solve real-life DMPs.

BFS is a FS extension with a membership degree range that differs from the previous extensions. Lee (2000) pioneered the BFS as an FS extension. Many academics have been interested in the merging of BFS and HFS in recent years, and good findings have been obtained. The concepts of hesitant bipolar-valued fuzzy sets (HBFSs) and their applications in DMPs were described by Han, Lou, and Chen (2016). Wei, Alsaadi, Hayat, and Alsaedi (2017) developed some hesitant bipolar fuzzy aggregation functions in MCDM, and Xu and Wei (2017) suggested some dual hesitant bipolar fuzzy aggregation functions in MCDM. The concepts of BFSs and their applications in DMPs were introduced by Abdullaha, Aslamb, and Ullaha (2014), and the concept of bipolar-valued hesitant fuzzy sets (BHFSs) with applications in MCDM was proposed by Ullah, Mahmood, Jan, Broumi, and Khan (2018). In a bipolar fuzzy environment, Alghamdi, Alshehri, and Akram (2018) proposed an enhanced method for MCDM. Following that, Multiple-attribute decision-making ELECTRE II approach under a bipolar fuzzy model was introduced by Shumaiza, Akram, and Al-Kenani (2019). More information on the outcomes of HBFSs and BHFSs, as well as their applications in multi criteria group decision-making (MCGDM), was explored by Mandal and Ranadive (2019). Later on, in 2020, Akram, Shumaiza, and Al-Kenani (2020) proposed an innovative MCGDM method for choosing green suppliers using a bipolar fuzzy PROMETHEE process. Additionally, Akram Shumaiza and Arshad (2020) created two novel techniques for diagnosing bipolar disorder: the bipolar fuzzy TOPSIS method and the bipolar fuzzy ELECTREI method. Bipolar fuzzy soft D-metric spaces were researched by Dalklç and Demirtaş (2021) in 2021. The concepts of HBFSSs were established by Wang, Wang, and Liu (2020), who also proposed a DMM (Wang-method) based on HBFSS that makes use of a scoring function and choice value in order to use HBFSs and SSs more effectively to address the uncertainty issues that are present in most real-world problems. A new MAGDM approach with 2-tuple linguistic bipolar fuzzy Heronian mean operators was recently introduced by Naz, Akram, Al-Shamiri, Khalaf, and Yousaf (2022), and Akram, Shumaiza, and Alcantud (2023) provided an efficient MCDM method with BFSs to resolve real-world DMPs.

In this study, we provide a new concept of WHBFSS and look at some of its fundamental properties in depth. We've also defined some novel notions of RMSDO, RMSDSM, and weighted score, and using these novel concepts, we have proposed an advanced and adjustable DMM for solving HBFSS and WHBFSS based DMPs. The following is the structure of this paper: The essential concepts and conclusions of FS, SS, FSS, HFS, HFSS, HBFS, and HBFSS are presented in Section 2, which will be important in later discussion. In Section 3, we provide a new concept of WHBFSS and look at some of its fundamental properties in detail. Also, we've defined some novel notions of RMSDO, RMSDSM, and weighted score, and using these novel concepts we have proposed an advanced and adjustable DMM for solving HBFSS and WHBFSS based DMPs. In Section 4, we show one real-life example to demonstrate the validity of our technique, and in Section 5, we provide a comparative analysis with an existing method. Finally, in Section 6, we bring the paper to a conclusion and discuss our future work.

2. Preliminaries

Let us consider Ω representing the starting universe and Q representing a nonempty set of parameters. Let the power set of Ω be denoted by P(Ω) and T⊆ Q.

Definition 2.1 (Zadeh, 1965) A FS Z on Ω is a set with a structure $Z=\{(\beta,\, \mu_Z(\beta)):\beta\in\Omega\}$, where the real-valued function $\mu_z:\Omega\to[0,1]$ is said to be the membership function and $\mu_z(\beta)$ is called the degree of membership for each object $\beta\in\Omega$. Assume that, FS(Ω) means the collection of all FSs on Ω.

Definition 2.2 (Molodtsov, 1999) A SS over the nonempty universe Ω is a pair (ψ,T), where ψ is a mapping defined by ψ: T→P (Ω).

Definition 2.3 (Maji, 2001) A pair (ψ,T) is said to be an FSS over Ω, where ψ: T→FS(Ω) is a mapping such that $\forall t\in T,\ \psi(t)=\{<\beta,\ \mu_{\psi(t)}(\beta)>:\beta\in\Omega\}$.

Assume that FSS(Ω) means the collection of all FSSs on Ω.

Definition 2.4 (Feng, Jun, Liu, and Li, 2010) Let $\Theta=(\psi,T)$ be a FSS on Ω and $\lambda(T)$ = {$\lambda(t)$: t∈T} be a threshold vector on T; then the λ(T)-level soft set (λ(T)-LSS) is denoted by L(Θ,λ)= (ψ_λ,T), and defined as $\psi_\lambda(t)=\{\beta\in\Omega:\mu_{\psi(t)}(\beta)\geq\lambda(t)\}$, $\forall t\in T$.

Definition 2.5 (Lee, 2000) A bipolar-valued fuzzy set (BFS) B on Ω is a set with a structure $B=\{(\beta,\ \mu_B^+(\beta),\mu_B^-(\beta)):\beta\in\Omega\}$, where $\mu_B^+:\Omega\to[0,1]$, and $\mu_B^-:\Omega\to[-1,0]$ are mappings, such that $\mu_B^+(\beta)$ means the positive information and $h_B^-(\beta)$ means the negative information $\forall\beta\in\Omega$.

As a matter of convenience, all BFSs on Ω are abbreviated as BFS(Ω).

Definition 2.6 (Abdullaha, 2014) A pair (ψ,T) is said to be a BFSS on Ω, where ψ: T⟶BFS(Ω) is a mapping such that $\forall t\in T,\ \psi(t)=\{(\beta,\ \mu_{\psi(t)}^+(\beta),\mu_{\psi(t)}^-(\beta)):\beta\in\Omega\}$.

Definition 2.7 (Torra, 2010) A HFS on Ω is denoted by $Z=\{\langle\beta,h_Z(\beta)\rangle:\beta\in\Omega\}$ and defined by the terms $h_Z(\beta)$ when applied to Ω, where $h_Z(\beta)$ is a collection of some various values in [0, 1], reflecting the possible membership degrees $\forall\beta\in\Omega$, and $h_Z(\beta)$ is called HFE.

Assume that HFS(Ω) means the collection of all HFSs on Ω.

Definition 2.8 (Wang, Li, & Chen, 2014) A pair (ψ,T) is said to be an HFSS over Ω, where ψ*: T→HFS(Ω)* is a mapping.

Definition 2.9 (Mandal, & Ranadive, 2019) A HBFS B on Ω is a set with a structure $B=\{<\beta,\ \mathrm{H}_B(\beta)=(h_B^+(\beta),h_B^-(\beta))>:\beta\in\Omega\}$, where $h_B^+(\beta)$ called the hesitant fuzzy positive element, is a set of some values in [0,1] denoting the possible satisfaction degree of $\beta\in\Omega$ to the corresponding property to the B; $h_B^-(\beta)$, called the hesitant fuzzy negative element, is a set of some values in [−1,0] denoting the negative satisfaction degree of $\beta\in\Omega$ to the opposite property to the B; and $\mathrm{H}_B(\beta)$, called the hesitant bipolar-valued fuzzy element (simply, HBFE), is a set of some values in [0,1]×[−1,0] to the B.

Simply, all HBFSs on Ω are abbreviated as HBFS(Ω), and HBFE(Ω) means all HBFEs in Ω.

Definition 2.10 (Wang, Wang, & Liu, 2020) A pair (ψ,T) is known as a HBFSS on Ω, where ψ is a mapping given by ψ:T⟶HBFS(Ω) and $\forall t\in T,\ \psi(t)=\{<\beta,\ \mathrm{H}_{\psi(t)}(\beta)>:\beta\in\Omega\}$.

Assume that HBFSS(Ω) means the collection of all HBFSSs on Ω.

Example 2.11 Let $\Omega=\{\beta_1,\beta_2\}$ be the set of the universe and $T=\{t_1,t_2,t_3\}$ be the set of parameters. Then

$$(\psi,\mathrm{T})=\left\{\begin{array}{l}\langle\ \mathrm{t}_1,((\beta_1,(\{0.7,0.6,\},\{-0.3,-0.4,-0.5\})),(\beta_2,(\{0.6,0.5,0.4\},\{-0.5,-0.7\})))\rangle,\\ \langle\ \mathrm{t}_2,((\beta_1,(\{0.8,0.7,0.6\},\ \{-0.4,-0.6\})),(\beta_2,(\{0.4,0.2\},\ \{-0.5,-0.6,-0.7\})))\rangle,\\ \langle\ \mathrm{t}_3,((\beta_1,(\{0.7,0.5,0.3\},\ \{-0.4,-0.8\})),(\beta_2,(\{0.8,0.6\},\ \{-0.3,-0.9\})))\rangle\end{array}\right\}$$

is an HBFSS on Ω.

Definition 2.12 (Wang, Wang, & Liu, 2020) The FSS (ψ_s,T) is called the score matrix of the HBFSS (ψ, T), where the scoring function of each member of the HBFS $\psi(t),\ \forall t\in T$ is the membership value of each member of the FS $\psi_s(t)$.

Wang-method (Wang, Wang, & Liu**,** 2020):

Algorithm 1:

Step 1. Enter the HBFSS (ψ, T).

Step 2. Compute the score matrix $\Theta=(\psi_s,T)$ associated with (ψ, T).

Step 3. The threshold vector $\lambda(T)$ can be obtained by calculating the average value allocated to each parameter.

Step 4. For each alternative, calculate the average-LSS $L(\Theta,\lambda)$ and the choice value C_j using the threshold vector λ.

Step 5. The best optimal choice is to select β_k if C_k is maximized.

Step 6. If β_k has many values, any of these β_k may be selected.

3. Weighted Hesitant Bipolar-Valued Fuzzy Soft Set and its Theoretical Analysis

Let Ω represent the starting universe, Q represent a set of parameters, and T, S, P⊆ Q.

Definition 3.1 A WHBFSS is a triple $\langle \psi, T, \sigma \rangle$, where (ψ_s, T) is an HBFSS over Ω and σ: T→[0,1] is a weight function that specifies the weight $\sigma_i = \sigma(t_i)$ for every $t_i \in T$. +++++++++++
We denote the collection of all *WHBFSSs* over Ω by *WHBFSS*(Ω).

Example 3.2 If we consider the HBFSS (ψ, T) as shown in Example 2.11, and assume that DMs has set the weight for the parameters in T as $\sigma_1 = \sigma(t_1) = 0.9;$ $\sigma_2 = \sigma(t_2) = 0.7;$ $\sigma_3 = \sigma(t_3) = 0.8,$ then we have the HBFSS (ψ, T) is changed into a WHBFSS $\langle \psi, T, \sigma \rangle$ as

$$\langle \psi, T, \sigma \rangle = \begin{Bmatrix} \langle (t_1, 0.9), ((\beta_1, (\{0.7,0.6,\}, \{-0.3,-0.4,-0.5\})), (\beta_2, (\{0.6,0.5,0.4\}, \{-0.5,-0.7\})))\rangle, \\ \langle (t_2, 0.7), ((\beta_1, (\{0.8,0.7,0.6\}, \{-0.4,-0.6\})), (\beta_2, (\{0.4,0.2\}, \{-0.5,-0.6,-0.7\})))\rangle, \\ \langle (t_3, 0.8), ((\beta_1, (\{0.7,0.5,0.3\}, \{-0.4,-0.8\})), (\beta_2, (\{0.8,0.6\}, \{-0.3,-0.9\})))\rangle \end{Bmatrix}$$

Definition 3.3 For two WHBFSSs $<\psi,T,\sigma>,<\varphi,S,\rho> \in WHBFSS(\Omega)$, we say that $<\psi,T,\sigma>$ is a sub-WHBFSS of $<\varphi,S,\rho>$ if (*i*). T ⊆ S and $\forall t \in T,\ \sigma(t) \le \rho(t),$ (*ii*). $\forall t \in T,\ \psi(t) \subseteq \varphi(t).$
We write $<\psi,T,\sigma> \tilde{\subseteq} <\varphi,S,\rho>.$

Definition 3.4 Let $<\psi,T,\sigma>,<\varphi,S,\rho> \in WHBFSS(\Omega)$. Then $<\psi,T,\sigma>$ and $<\varphi,S,\rho>$ are equal-sets, denoted by $<\psi,T,\sigma> \ = \ <\varphi,S,\rho>$ if and only if $\forall t \in T,\ \sigma(t) = \rho(t)$ and $\psi(t) = \varphi(t).$

Proposition 3.5 Let $<\psi,T,\sigma>,<\varphi,S,\rho>,<\tau,P,\delta> \in WHBFSS(\Omega)$. Then
[*i*]. $<\psi,T,\sigma>=<\varphi,S,\rho>$ and $<\varphi,S,\rho>=<\tau,P,\delta> \ \Rightarrow <\psi,T,\sigma>=<\tau,P,\delta>$
[*ii*]. $<\psi,T,\sigma> \tilde{\subseteq} <\varphi,S,\rho>$ and $<\varphi,S,\rho> \tilde{\subseteq} <\tau,P,\delta> \ \Rightarrow <\psi,T,\sigma> \tilde{\subseteq} <\tau,P,\delta>$
[*ii*]. $<\psi,T,\sigma> \tilde{\subseteq} <\varphi,S,\rho>$ and $<\varphi,S,\rho> \tilde{\subseteq} <\psi,T,\sigma> \ \Rightarrow <\psi,T,\sigma>=<\varphi,S,\rho>$

Definition 3.6 The complement of a *WHBFSS* $<\psi,T,\sigma> \in WHBFSS(\Omega)$ denoted by $<\psi,T,\sigma>^C$ is defined by $<\psi,T,\sigma>^C = <\psi^C,T,\sigma^C>,$ where $\psi^C : T \to HBFS(\Omega)$ and $\sigma^C : T \to [0,1]$ are functions given by $\forall t \in T,\ \psi^C(t) = (\psi(t))^C$ and $\sigma^C(t) = 1 - \sigma(t).$

Proposition 3.7 Let $<\psi,T,\sigma> \in WHBFSS(\Omega)$, then $\left(<\psi,T,\sigma>^C\right)^C = <\psi,T,\sigma>.$

Definition 3.8 Union between two WHBFSSs $<\psi,T,\sigma>,<\varphi,S,\rho> \in WHBFSS(\Omega)$ is denoted by $<\psi,T,\sigma> \tilde{\cup} <\varphi,S,\rho>$ and defined as $<\psi,T,\sigma> \tilde{\cup} <\varphi,S,\rho> = <\tau,P,\delta>,$ where $P = T \cup S$ and

$$\tau(t) = \begin{cases} \psi(t), & \text{if } t \in T \\ \varphi(t), & \text{if } t \in S \\ \psi(t) \cup \varphi(t), & \text{if } t \in T \cap S \end{cases} \quad \text{and} \quad \delta(t) = \begin{cases} \sigma(t), & \text{if } t \in T \\ \rho(t), & \text{if } t \in S \\ \max\{\sigma(t), \rho(t)\}, & \text{if } t \in T \cap S. \end{cases}.$$

Definition 3.9 Intersection between two WHBFSSs $<\psi,T,\sigma>,<\varphi,S,\rho> \in WHBFSS(\Omega)$ is denoted by $<\psi,T,\sigma> \tilde{\cap} <\varphi,S,\rho>$ and defined as $<\psi,T,\sigma> \tilde{\cap} <\varphi,S,\rho> = <\tau,P,\delta>,$ where $P = T \cup S$ and

$$\tau(t) = \begin{cases} \psi(t), & \text{if } t \in T \\ \varphi(t), & \text{if } t \in S \\ \psi(t) \cap \varphi(t), & \text{if } t \in T \cap S \end{cases} \quad \text{and} \quad \delta(t) = \begin{cases} \sigma(t), & \text{if } t \in T \\ \rho(t), & \text{if } t \in S \\ \min\{\sigma(t), \rho(t)\}, & \text{if } t \in T \cap S. \end{cases}.$$

Proposition 3.10 Let us consider $<\psi,T,\sigma>,<\varphi,S,\rho>,<\tau,P,\delta> \in WHBFSS(\Omega)$, then
[i] Associative Laws

$$<\psi,T,\sigma> \tilde{\cup} \left(<\varphi,S,\rho> \tilde{\cup} <\tau,P,\delta>\right) = \left(<\psi,T,\sigma> \tilde{\cup} <\varphi,S,\rho>\right) \tilde{\cup} <\tau,P,\delta>$$

$$<\psi,T,\sigma> \tilde{\cap} \left(<\varphi,S,\rho> \tilde{\cap} <\tau,P,\delta>\right) = \left(<\psi,T,\sigma> \tilde{\cap} <\varphi,S,\rho>\right) \tilde{\cap} <\tau,P,\delta>$$

[ii] Distributive Laws

$$<\psi,T,\sigma>\tilde{\cap}\left(<\varphi,S,\rho>\tilde{\cup}<\tau,P,\delta>\right)=\left(<\psi,T,\sigma>\tilde{\cap}<\varphi,S,\rho>\right)\tilde{\cup}\left(<\psi,T,\sigma>\tilde{\cap}<\tau,P,\delta>\right)$$

$$<\psi,T,\sigma>\tilde{\cup}\left(<\varphi,S,\rho>\tilde{\cap}<\tau,P,\delta>\right)=\left(<\psi,T,\sigma>\tilde{\cup}<\varphi,S,\rho>\right)\tilde{\cap}\left(<\psi,T,\sigma>\tilde{\cup}<\tau,P,\delta>\right)$$

[iii] De Morgan's Laws

$$\left(<\psi,T,\sigma>\tilde{\cap}<\varphi,S,\rho>\right)^{C}=<\psi,T,\sigma>^{C}\tilde{\cup}<\varphi,S,\rho>^{C}$$

$$\left(<\psi,T,\sigma>\tilde{\cup}<\varphi,S,\rho>\right)^{C}=<\psi,T,\sigma>^{C}\tilde{\cap}<\varphi,S,\rho>^{C}$$

Definition 3.11 The RMSDO $\Delta: HBFE(\Omega)\rightarrow[-1,1]$ defined by $\forall \mathrm{H}(\beta)=(h^{+}(\beta),h^{-}(\beta))\in HBFE(\Omega)$,

$$\Delta(\mathrm{H}(\beta))=\left(\frac{1}{\left|h^{+}(\beta)\right|}\sum_{\mu^{+}\in h^{+}(\beta)}\left(\mu^{+}\right)^{2}\right)^{\frac{1}{2}}-\left(\frac{1}{\left|h^{-}(\beta)\right|}\sum_{\mu^{-}\in h^{-}(\beta)}\left(\mu^{-}\right)^{2}\right)^{\frac{1}{2}}.$$

Example 3.12 If we consider the WHBFSS $\langle\psi,T,\sigma\rangle$ as shown in Example 3.2, then we have the set of all HBFEs in Ω,

HBFE(Ω) = { ({0.7,0.6,},{-0.3,-0.4,-0.5}), ({0.8,0.7,0.6}, {-0.4,-0.6}),
({0.7,0.5,0.3}, {-0.4,-0.8}), ({0.6,0.5,0.4}, {-0.5,-0.7}),
({0.4,0.2}, {-0.5,-0.6,-0.7}), ({0.8,0.6}, {-0.3,-0.9})}.

Then

$$\Delta\left(\{0.7,0.6,\},\{-0.3,-0.4,-0.5\}\right)$$

$$=\left(\frac{1}{2}\left\{(0.7)^{2}+(0.6)^{2}\right\}\right)^{\frac{1}{2}}-\left(\frac{1}{3}\left\{(-0.3)^{2}+(-0.4)^{2}+(-0.5)^{2}\right\}\right)^{\frac{1}{2}}=0.24367.$$

Similarly, we have

$$\Delta\left(\{0.8,0.7,0.6\},\ \{-0.4,-0.6\}\right)=0.19485,$$
$$\Delta\left(\{0.7,0.5,0.3\},\ \{-0.4,-0.8\}\right)=-0.10647,$$
$$\Delta\left(\{0.6,0.5,0.4\},\ \{-0.5,-0.7\}\right)=-0.10165,$$
$$\Delta\left(\{0.4,0.2\},\{-0.5,-0.6,-0.7\}\right)=-0.2893,$$
$$\Delta\left(\{0.8,0.6\},\ \{-0.3,-0.9\}\right)=-0.036287.$$

Definition 3.13 The RMSDSM of the HBFSS $<\psi,T,\sigma>\in WHBFSS(\Omega)$ is denoted by $<\psi_{\Delta},T,\sigma>$ and defined as $\forall t\in T,\ \psi_{\Delta}(t)=\{<\beta,\ \Delta(\mathrm{H}_{\psi(t)}(\beta))>:\beta\in\Omega\}$.

Example 3.14 If we consider the WHBFSS $\langle\psi,T,\sigma\rangle$ as shown in Example 3.2, then we have

$$\psi_{\Delta}(t_1)=\left\{<\beta_1,0.24367>,<\beta_2,-0.10165>\right\},$$
$$\psi_{\Delta}(t_2)=\left\{<\beta_1,0.19485>,<\beta_2,-0.2893>\right\},$$
$$\psi_{\Delta}(t_3)=\left\{<\beta_1,-0.10647>,<\beta_2,-0.03629>\right\}.$$

Now, we present our advanced machine learning algorithm for solving DMPs based on HBFSS and WHBFSS. The steps of our proposed DMM are listed below:

Algorithm 2

Step 1: Enter a nonempty universe $\Omega=\left\{\beta_1,\beta_2,...,\beta_n\right\}$, a set of parameters $T=\left\{t_1,t_2,...,t_m\right\}$, and a group of experts $\left\{DM_1,DM_2,...,DM_q\right\}$.

Step 2: Enter the BFSSs $(\psi_1,T),\ (\psi_2,T),...,(\psi_q,T)$, as provided by each expert.

Step 3: Compute the resultant HBFSS (ψ,T) from the BFSSs $(\psi_1,T),\ (\psi_2,T),...,(\psi_q,T)$

Step 4: Enter a weight σ corresponding to the HBFSS (ψ,T), where $\sigma:T\rightarrow[0,1]$

Step 5: Obtain the WHBFSS $\langle\psi,T,\sigma\rangle$ with regards to the weight σ.

Step 6: Compute the RMSDSM $\langle\psi_{\Delta},T,\sigma\rangle$ and the weighted score using the formula

$$\varpi_j=\frac{1}{m}\sum_{k=1}^{m}\sigma(t_k)\times\Delta(\mathrm{H}_{\psi(t_k)}(\beta_j)),\ \forall\beta_j\in\Omega$$

Step 7: The best optimal choice is to select β_s if ϖ_s is maximized.

Step 8. If β_s has many values, any of these β_s may be selected.

4. Results and Discussion

In this section, a real-life DMP is provided to demonstrate the validity of our suggested method.

Step 1: Let $\Omega = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5\}$ be the set of manuscripts submitted in the conference for the best manuscript award, and $T = \{t_1, t_2, t_3, t_4, t_5\}$ be the parameters set, where

$T = \{t_1 =$ invention and originality,
$t_2 =$ significance of outcomes,
$t_3 =$ applications,
$t_4 =$ modernity of references,
$t_5 =$ precision of language and clarity of goal$\}$.

Step 2: Suppose the manuscripts are reviewed by three experts, and their observations $(\psi, T), (\varphi, T)$ and (ρ, T) are in Tables 1, 2, and 3, respectively.

Step 3: Based on the results of combining the three experts' observations, we have the resultant HBFSS (Ψ, T) shown in Table 4.

Step 4: Assume that the decision maker has set the weights for the parameters in T as follows: $\sigma_1 = \sigma(t_1) = 0.9;\ \ \sigma_2 = \sigma(t_2) = 0.7;\ \ \sigma_3 = \sigma(t_3) = 0.9;\ \ \sigma_4 = \sigma(t_4) = 0.8;\ \ \sigma_5 = \sigma(t_5) = 0.5.$

Step 5: Then we have a weighted function σ for HBFSS (Ψ, T), where σ: T→[0,1] and the HBFSS (Ψ, T) is changed into a WHBFSS $\langle \Psi, T, \sigma \rangle$ as in Table 5.

Step 6: We obtain the RMSDSM $< \Psi_\Delta, T, \sigma >,$ whose tabular representation is in Table 6. Table 6 shows the results of the computation of the weighted score ϖ_j at step 6.

Step 7: From the last column in Table 6, we have the optimal choice β_4.

Table 1. The BFSS (ψ, T)

Ω	t_1	t_2	t_3	t_4	t_5
β_1	0.7,-0.3	0.6,-0.4	0.5,-0.8	0.8,-0.4	0.2,-0.7
β_2	0.6,-0.5	0.2,-0.5	0.7,-0.9	0.6,-0.9	0.4,-0.6
β_3	0.7,-0.2	0.4,-0.6	0.7,-0.5	0.8,-0.4	0.3,-0.7
β_4	0.8,-0.4	0.5,-0.3	0.6,-0.4	0.8,-0.2	0.2,-0.8
β_5	0.6,-0.7	0.3,-0.4	0.6,-0.8	0.7,-0.2	0.4,-0.7

Table 2. The BFSS (φ, T)

Ω	t_1	t_2	t_3	t_4	t_5
β_1	0.6,-0.4	0.8,-0.6	0.3,-0.4	0.6,-0.5	0.4,-0.5
β_2	0.4,-0.5	0.4,-0.7	0.6,-0.3	0.5,-0.7	0.5,-0.4
β_3	0.5,-0.4	0.9,-0.8	0.5,-0.3	0.4,-0.6	0.6,-0.7
β_4	0.8,-0.6	0.7,-0.7	0.5,-0.2	0.7,-0.6	0.7,-0.7
β_5	0.6,-0.8	0.5,-0.8	0.4,-0.6	0.5,-0.6	0.6,-0.5

Table 3. The BFSS (ρ, T)

Ω	t_1	t_2	t_3	t_4	t_5
β_1	0.5,-0.5	0.7,-0.4	0.7,-0.4	0.6,-0.9	0.6,-0.3
β_2	0.5,-0.7	0.3,-0.6	0.8,-0.6	0.4,-0.5	0.9,-0.2
β_3	0.3,-0.6	0.5,-0.7	0.7,-0.3	0.3,-0.8	0.9,-0.5
β_4	0.6,-0.5	0.5,-0.5	0.7,-0.3	0.3,-0.2	0.9,-0.6
β_5	0.5,-0.7	0.5,-0.6	0.6,-0.7	0.3,-0.6	0.8,-0.3

Table 4. The HBFSS (Ψ, T)

Ω	t_1	t_2	t_3	t_4	t_5
β_1	{0.7,0.6,0.5} {-0.3,-0.4,-0.5}	{0.8,0.7,0.6} {-0.4,-0.6}	{0.7,0.5,0.3} {-0.4,-0.8}	{0.8,0.6} {-0.4,-0.5,-0.9}	{0.6,0.4,0.2} {-0.3,-0.5,-0.7}
β_2	{0.6,0.5,0.4} {-0.5,-0.7}	{0.4,0.3,0.2} {-0.5,-0.6,-0.7}	{0.8,0.7,0.6} {-0.3,-0.6,-0.9}	{0.6,0.5,0.4} {-0.5,-0.7,-0.9}	{0.9,0.5,0.4} {-0.2,-0.4,-0.6}
β_3	{0.7,0.5,0.3} {-0.2,-0.4,-0.6}	{0.9,0.5,0.4} {-0.6,-0.7,-0.8}	{0.7,0.5} {-0.3,-0.5}	{0.8,0.4,0.3} {-0.4,-0.6,-0.8}	{0.9,0.6,0.3} {-0.5,-0.7}
β_4	{0.8,0.6} {-0.4,-0.5,-0.6}	{0.7,0.5} {-0.3,-0.5,-0.7}	{0.7,0.6,0.5} {-0.2,-0.3,-0.4}	{0.8,0.7,0.3} {-0.2,-0.6}	{0.9,0.7,0.2} {-0.6,-0.7,-0.8}
β_5	{0.6,0.5} {-0.7,-0.8}	{0.5,0.3} {-0.4,-0.6,-0.8}	{0.6,0.4} {-0.6,-0.7,-0.8}	{0.7,0.5,0.3} {-0.2,-0.6}	{0.8,0.6,0.4} {-0.3,-0.5,-0.7}

Table 5. The WHBFSS $\langle \Psi, T, \sigma \rangle$

Ω	t_1 0.9	t_2 0.7	t_3 0.9	t_4 0.8	t_5 0.5
β_1	{0.7,0.6,0.5} {-0.3,-0.4,-0.5}	{0.8,0.7,0.6} {-0.4,-0.6}	{0.7,0.5,0.3} {-0.4,-0.8}	{0.8,0.6} {-0.4,-0.5,-0.9}	{0.6,0.4,0.2} {-0.3,-0.5,-0.7}
β_2	{0.6,0.5,0.4} {-0.5,-0.7}	{0.4,0.3,0.2} {-0.5,-0.6,-0.7}	{0.8,0.7,0.6} {-0.3,-0.6,-0.9}	{0.6,0.5,0.4} {-0.9,-0.7,-0.5}	{0.9,0.5,0.4} {-0.2,-0.4,-0.6}
β_3	{0.7,0.5,0.3} {-0.2,-0.4,-0.6}	{0.9,0.5,0.4} {-0.6,-0.7,-0.8}	{0.7,0.5} {-0.3,-0.5}	{0.8,0.4,0.3} {-0.4,-0.6,-0.8}	{0.9,0.6,0.3} {-0.5,-0.7}
β_4	{0.8,0.6} {-0.4,-0.5,-0.6}	{0.7,0.5} {-0.3,-0.5,-0.7}	{0.7,0.6,0.5} {-0.2,-0.3,-0.4}	{0.8,0.7,0.3} {-0.2,-0.6}	{0.9,0.7,0.2} {-0.6,-0.7,-0.8}
β_5	{0.6,0.5} {-0.7,-0.8}	{0.5,0.3} {-0.4,-0.6,-0.8}	{0.6,0.4} {-0.6,-0.7,-0.8}	{0.7,0.5,0.3} {-0.2,-0.6}	{0.8,0.6,0.4} {-0.3,-0.5,-0.7}

Table 6. The RMSDSM $< \Psi_\Delta, T, \sigma >$, with weighted score ϖ_j

Ω	t_1 0.9	t_2 0.7	t_3 0.9	t_4 0.8	t_5 0.5	ϖ_j
β_1	0.197282	0.194847	-0.106465	0.069403	-0.093942	0.04533592
β_2	-0.101653	-0.294617	0.056675	-0.212172	0.205655	-0.06272444
β_3	0.093942	-0.067045	0.195965	-0.077155	0.039798	0.03443196
β_4	0.200484	0.082285	0.294617	0.190490	-0.036418	0.12747468
β_5	-0.199397	-0.209514	-0.194847	0.078777	0.095834	-0.07810816

Advantages: When we use Algorithm 2, we get fewer object choices, which makes it easier for us to make a decision. However, by using Algorithm 2, we can obtain more detailed data, which will assist leaders in making decisions. If there are lots of optimal choices to be selected in the 7th step, we can return to the 4th step and adjust the weight so that we can find one single optimal solution.

5. Comparison Analysis

In the following, we have to show that the Wang method is not sufficient to solve HBFSS based DMPs. Let $\Omega = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5\}$ be the set of manuscripts submitted to the conference for the best manuscript award, and $T = \{t_1, t_2, t_3, t_4, t_5\}$ be the parameters set where

$T = \{t_1 =$ invention and originality,

$t_2 =$ significance of outcomes,

$t_3 =$ applications,

$t_4 =$ modernity of references,

$t_5 =$ precision of language and clarity of goal$\}$

Suppose the manuscripts are reviewed by three experts, and their observations $(\psi, T), (\varphi, T)$ and (ρ, T) are in Tables 1, 2, and 3, respectively.

Step 1: After combining the three experts' observations, we have the resultant HBFSS (Ψ, T) as shown in Table 4.

Step 2: We obtain the score matrix $\Theta=(\Psi_s,T)$ associated with (Ψ,T), shown in Table 7.

Step 3: Calculate the average value assigned to each parameter, that is,

$$\lambda(t_1)=\frac{1}{5}\sum_{k=1}^{5}\mu_{\psi_s(t_1)}(\beta_k)=0.55,$$

$$\lambda(t_2)=\frac{1}{5}\sum_{k=1}^{5}\mu_{\psi_s(t_2)}(\beta_k)=0.55,$$

$$\lambda(t_3)=\frac{1}{5}\sum_{k=1}^{5}\mu_{\psi_s(t_3)}(\beta_k)=0.55,$$

$$\lambda(t_4)=\frac{1}{5}\sum_{k=1}^{5}\mu_{\psi_s(t_4)}(\beta_k)=0.55,$$

$$\lambda(t_5)=\frac{1}{5}\sum_{k=1}^{5}\mu_{\psi_s(t_5)}(\beta_k)=0.55.$$

Then, $\lambda(T)=\{\lambda(t_1),\ \lambda(t_2),\ \lambda(t_3),\ \lambda(t_4),\ \lambda(t_5)\}=\{0.55,\ 0.55,\ 0.55,\ 0.55,\ 0.55\}$.

Step 4: Calculate the average-LSS $L(\Theta,\lambda)$ and the choice value C_j, shown in Table 8.

Step 5: From Table 8, we see that here all the choice values are the same, namely 3, so in this case the decision maker cannot choose the optimal decision.

Table 7. The score matrix $\Theta=(\Psi_s,T)$ associated to (Ψ,T)

Ω	t_1	t_2	t_3	t_4	t_5
β_1	0.50	0.60	0.55	0.65	0.45
β_2	0.55	0.45	0.65	0.60	0.50
β_3	0.45	0.65	0.50	0.55	0.60
β_4	0.60	0.55	0.45	0.50	0.65
β_5	0.65	0.50	0.60	0.45	0.55

Table 8. The average-LSS $L(\Theta,\lambda)$, with choice value C_j

Ω	s_1	s_2	s_3	s_4	s_5	C_j
β_1	0	1	1	1	0	3
β_2	1	0	1	1	0	3
β_3	0	1	0	1	1	3
β_4	1	1	0	0	1	3
β_5	1	0	1	0	1	3

This is enough to prove that the Wang-method is not sufficient to solve HBFSS based DMPs, but the constructed method in this paper is very advantageous to solving these HBFSS based DMPs (Section 4). The novelty of our proposed DMM is the concept of the RMSDO rather than the score function, which makes our DMM more stable and more feasible than the Wang-method and another difference is the concept of the weighted score rather than the choice value, which makes our DMM adjustable.

6. Conclusions

In this paper, we have introduced the new concept of WHBFSS and studied some basic operations on it in detail. In addition, we have defined some novel notions of RMSDO, RMSDSM, and weighted score, and using these novel notations, we have proposed an advanced and adjustable DMM for solving HBFSS and WHBFSS based DMPs. The novelty of our proposed DMM is the concept of the RMSDO rather than the score function, which makes our DMM more stable and more feasible than the existing method. Algorithm 2 is more suitable for many real-world applications because of this adjustable feature. We can see that it can be related to a variety of fields that have dubious relations by means of types of operations. The approach should be expanded in the future to address relevant issues such as computer science, software engineering, current life condition, and so on.

In a future study, we will give more broad properties and operations on WHBFSSs and extend this proposed DMM to other real-life applications in the fields of pattern recognition and medical diagnostics.

Abbreviations

BFS	Bipolar-valued fuzzy set
BFSS	Bipolar-valued fuzzy soft set
BHFS	Bipolar-valued hesitant fuzzy set
DMM	Decision-making method
DMP	Decision-making problem
FS	Fuzzy set
FSS	Fuzzy soft set
HF	Hesitant fuzzy
HFS	Hesitant fuzzy set
HBFSS	Hesitant fuzzy soft set
HBFSS	Hesitant bipolar-valued fuzzy set
HBFSS	Hesitant bipolar-valued fuzzy soft set
IFS	Intuitionistic fuzzy set
LSS	Level soft set
MCDM	Multi criteria decision-making
MCGDM	Multi criteria group decision-making
RMSDO	Root mean square difference operator
RMSDSM	Root mean square difference score matrix
SS	Soft set

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