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Original Article

A numerical approach to the solution of nonlinear Volterra-Fredholm integro-differential equations

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Abstract

In this study, a numerical method is presented for solving first-order Volterra-Fredholm Integro-differential Equations (V-FIDEs). The approach converts V-FIDEs into integral equations and uses power series polynomials to approximate them. The modeled problem is transformed into a system of algebraic equations, and then it is solved using the standard collocation method. Numerical examples are utilized to assess the method's effectiveness after the approach's uniqueness and convergence have been established. The results demonstrate that the method competes favorably with other methods.

Keywords: collocation method, Volterra-Fredholm, integro-differential equations, approximate solution, polynomial power series

1. Introduction

Integro-differential Equations (IDEs) are strong tools in pure and applied mathematics, engineering, and physics. IDEs are used in many mathematical representations of physical phenomena in fluid dynamics, heat transfer, diffusion processes, neutron diffusion, biological models, nano-hydrodynamics, economics, and population growth models. In 1926, Vito Volterra used an integro-differential approach to explore population increase with a focus on hereditary effects (Rahman, 2007).

The various methods for finding the numerical solution of IDEs include Bernstein method (Irfan, Kumar, & Kapoor, 2014), Adomian decompositions method (Khan & Bakoda, 2013; Mittal & Nigam, 2008), Finite difference-Simpson method (Garba & Bichi, 2021), collocation method (Agbolade & Anake, 2017; Ajileye & Aminu, 2022; Ajileye & Amoo, 2023; Ajileye, James, Ayinde & Oyedepo 2022; Nemati, Lima & Ordok-hani, 2015), the Modified Adomian Decomposition Method (MADM) (Ahmad, Ullah, Ullah, Ahmad, Shah & Ahmad, 2021), hybrid linear multistep method (Mehdiyera, Ibrahim & Imanova, 2019; Mehdiyera, Imanova & Ibrahim, 2015), Chebyshev-Galerkin method (Issa & Saleh, 2017), Bernoulli matrix method (Bhraway, Tohidi, & Soleymani, 2012), differential transform method (Ercan & Kharerah, 2013), Haar collocation method (Amin, Shah, Mlaiki, Yuzbasi, Abdeljawad, & Hussain, 2022), Lagrange interpolation (Shahsavaran & Shahsavaran, 2012), differential transformation (Darania & Ebadian, 2007), block pulse functions' operational matrices (Rahmani, Rahimin, & Mordad, 2011), Chebyshev polynomials (Maadadi & Rahmoune, 2018), Optimal Auxiliary Function Method (OAFM) (Zada, Al-Hamami, Nawaz, Jehanzeb, Morsy, Abdei-Aty, & Nisar, 2021), and spectral homotopy method (Atabakan, Nasab, Kilicman, & Eshkuvatov, 2013). Hou, Niu, Xu & Ngolo (2021) presented a method combining the simplified reproducing kernel method and the homotopy perturbation method to solve nonlinear Volterra-Fredholm integro-differential equations. They converted nonlinear to linear using

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the homotopy perturbation method and adopted the simplified reproducing kernel method to solve the linear problem.

In this paper, we present a collocation method for the numerical solution of first-order V-FIDEs of the form:

$$y'(x) + q(x)y(x) + \lambda_1 \int_0^x k_1(x,t)\, F_1(y(t))dt + \lambda_2 \int_0^1 k_2(x,t)\, F_2(y(t))dt = g(x) \tag{1}$$

with the initial condition,

$$y(a) = c \tag{2}$$

where $k_1(x,t)$ and $k_2(x,t)$ are the Volterra and Fredholm integral kernel functions, respectively, and $\lambda^1, \lambda^2, a, c, and\ q$ are known constants. $g(x)$ is a given function, and $y(x)$ is the unknown function to be determined.

2. Basic Definitions and Terms

We give certain definitions and fundamental notions in this section for the purpose of problem formulation.

Definition 1. (Ajileye & Amoo, 2023) Let $(a_m), m \geq 0$ be a sequence of real numbers. The power series w with coefficients a_m is an expression.

$$y(w) = \sum_{m=0}^{M} a_m w^m = \emptyset(w)\boldsymbol{A} \tag{3}$$

where

$$\emptyset(w) = [1 \;\; w \;\; w^2 \;\; \cdots \;\; w^M \,], \;\; \boldsymbol{A} = [a_0 \;\; a_1 \cdots a_M \,]^T$$

Definition 2. (Agbolade & Anake, 2017) The desired collocation points within an interval are determined using this method. For interval [a, b] they are provided by

$$l_u = a + \frac{(b-a)u}{M}, u = 0, 1, 2 \ldots \ldots M \tag{4}$$

Definition 3. (Ajileye, James, Ayinde, & Oyedepo, 2022) Let z(s) be an integrable function, then

$${}_0I_x^{\beta}(z(s)) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} z(s)ds \tag{5}$$

Definition 4. (Ajileye, James, Ayinde, & Oyedepo, 2022) Let $y(x)$ be a continuous function, then

$${}_0I_x^{\beta}\left(y^{(\beta)}(x)\right) = y(x) - \sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!} x^k \tag{6}$$

Definition 5. (Berinde, 2007) (Strict contraction) Let (X,d) be a metric space. A mapping $T: X \to X$ is a strict contraction if T is α −Lipschitzian with $\alpha\varepsilon[0,1)$.

$$d(T(x), T(y)) \leq \alpha d(x,y)\ \forall x, y\varepsilon X$$

Theorem 1. (Berinde, 2007) (Banach fixed point theorem) Let (X,d) be a complete metric space and T: X→X is a contraction on X, then T has a unique fixed point $x\varepsilon X$ such that $T(x) = x$, and moreover the Picard successive approximations iterative scheme converges, given by

$$x_n = Tx_{n-1}$$

3. Materials and Methods

In this section, we establish the uniqueness of the solution and implement a collocation approach for the numerical solution of V-FIDEs.

Lemma 1. (Integral form): Let $y\varepsilon\mathbb{C}\ ((0,1), \mathbb{R})$ be the solution to equation (1), then it is equivalent to

$$y(x) + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \big(q(x)y(x)\big)ds + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left(\lambda_1 \int_0^x k_1(x,t)\, F_1(y(t))dt + \lambda_2 \int_0^1 k_2(x,t)\, F_2(y(t))dt\right) ds = B(x) \tag{7}$$

where

$$B(x) = \sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!} x^k + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} g(s)ds$$

Proof.

Applying to equation (1) the operator ${}_0I_x^\alpha(.)$gives

$$
{}_0I_x^\alpha\left(y'(x)\right)+\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\left(q(x)y(x)\right)ds+
$$
$$
{}_0I_x^\alpha\left(\lambda_1\int_0^x k_1(x,t)\,F_1(y(t))dt+\lambda_2\int_0^1 k_2(x,t)\,F_2(y(t))dt\right)={}_0I_x^\alpha(g(x))
$$

and using equation (5) gives

$$
y(x)+\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\left(q(x)y(x)\right)ds+\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\left(\lambda^1\int_0^x k^1(x,t)F^1\left(y(t)\right)dt+\right.
$$
$$
\left.\lambda_2\int_0^1 k_2(x,t)\,F_2(y(t))dt\right)ds=\sum_{k=0}^N\frac{y^{(k)}(0)}{k!}x^k+\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}g(s)ds
$$
$$
y(x)+\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\left(q(x)y(x)\right)ds+\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\left(\lambda_1\int_0^x k_1(x,t)\,F_1(y(t))dt+\right.
$$
$$
\left.\lambda_2\int_0^1 k_2(x,t)\,F_2(y(t))dt\right)ds=W(x)
$$
$$
y(x)=W(x)-\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\left(q(x)y(x)\right)ds-\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\left(\lambda_1\int_0^x k_1(x,t)\,F_1(y(t))dt+\right.
$$
$$
\left.\lambda_2\int_0^1 k_2(x,t)\,F_2(y(t))dt\right)ds \tag{8}
$$

where

$$
W(x)=\sum_{k=0}^N\frac{y^{(k)}(0)}{k!}x^k+\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}g(s)ds
$$

3.1 Uniqueness of the method

In order to establish the method's uniqueness, we provide the following hypothesis:
H_1: There exist constants $L_1, L_2>0$, such that for any $y_1, y_2\in C([0,1],\mathbb{R})$

$$
|F_1(y_1)-F_1(y_2)|\le L_1|y_1-y_2|
$$
$$
and
$$
$$
|F_2(y_1)-F_2(y_2)|\le L_2|y_1-y_2|
$$

H_2: There exist two functions k_1^* and $k_2^*\in C([0,1]\times[0,1],\mathbb{R})$ in the set of all positive functions such that

$$
k_1^*=max_{x\in[0,1]}|\lambda_1|\int_0^x|k(x,t)|dt<\infty
$$
$$
k_2^*=max_{x\in[0,1]}|\lambda_2|\int_0^1|k(x,t)|dt<\infty
$$

H_3: The function $g\in\mathbb{R}$ is continuous such that

$$
q^*=max_{x\in[0,1]}|q(x)|
$$

Theorem 2. Let $T: X\to X$ be the mapping defined by equation (8). Then T is a strict contraction if

$$
\left(\frac{q^*+k_1^*+k_2^*}{\Gamma(\alpha+1)}\right)<1
$$

Proof.

Let $y_1(x), y_2(x)\varepsilon X$. Applying Banach fixed point theorem to equation (8)

$$
(Ty_1)(x)=W(x)-\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\left(q(x)y_1(x)\right)ds-\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\left(\lambda_1\int_0^x k_1(x,t)\,F_1(y_1(t))dt+\right.
$$
$$
\left.\lambda_2\int_0^1 k_2(x,t)\,F_2(y_1(t))dt\right)ds \tag{9}
$$

and

$$(Ty_2)(x) = W(x) - \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(q(x)y_1(x)\right)ds - \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(\lambda_1\int_0^x k_1(x,t)\,F_1(y_2(t))dt + \lambda_2\int_0^1 k_2(x,t)\,F_2(y_2(t))dt\right)ds \quad (10)$$

Subtracting equation (10) from equation (9) gives

$$(Ty_1)(x) - (Ty_2)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left[q(x)\left(y_2(x)-y_1(x)\right)\right]ds + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}$$
$$\left(\lambda_1\int_0^x k_1(x,t)\,F_1(y_2(t)-y_1(t))dt + \lambda_2\int_0^1 k_2(x,t)\,F_2(y_2(t)-y_1(t))dt\right)ds$$

Taking the absolute value of both sides gives

$$|(Ty_1)(x) - (Ty_2)(x)| = \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left|q(x)\left(y_2(x)-y_1(x)\right)\right|ds +$$
$$\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(|\lambda_1|\int_0^x |k_1(x,t)|\,|F_1(y_2(t)-y_1(t))|dt + |\lambda_2|\int_0^1 |k_2(x,t)|\,|F_2(y_2(t)-y_1(t))|dt\right)ds$$

Taking the maximum of both sides and applying $H_1 - H_3$ gives $d(Ty_1, Ty_2) \le \left(\frac{q^* + k_1^* + k_2^*}{\Gamma(\alpha+1)}\right) d(y_1, y_2)$
Therefore T is a strict contraction mapping.

Theorem 3. (Continuity) Let (X, d) be a metric space and $T: X \to X$ be a mapping, and let $y_n(x), y(x) \varepsilon X$ and the $\lim_{x\in[0,1]} y_n(x) = y(x)$. Then T is continuous if $d(Ty_n, Ty) \to 0 \; as \; n \to \infty$.

Proof.

$$|(Ty_n)(x) - (Ty)(x)| = \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left|q(x)\left(y(x)-y_n(x)\right)\right|ds +$$
$$\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(|\lambda_1|\int_0^x |k_1(x,t)|\,|F_1(y(t)-y_n(t))|dt + |\lambda_2|\int_0^1 |k_2(x,t)|\,|F_2(y(t)-y_n(t))|dt\right)ds$$
$$|(Ty_n)(x) - (Ty)(x)| \le \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\max_{x\in[0.1]}|q(x)|\max_{x\in[0.1]}\left|\left(y(x)-y_n(x)\right)\right|ds +$$
$$\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(|\lambda_1|\max_{x\in[0.1]}\int_0^x |k_1(x,t)|\max_{x\in[0.1]}|F_1(y(t)-y_n(t))|dt + |\lambda_2|\max_{x\in[0.1]}\int_0^1 |k_2(x,t)|\max_{x\in[0.1]}|F_2(y(t)-y_n(t))|dt\right)ds$$
$$d(Ty_1, Ty_2) \le \left(\frac{q^* + k_1^* + k_2^*}{\Gamma(\alpha+1)}\right) d(y_1, y_2)$$

Since $d(y_n, y) \to 0 \; as \; n \to \infty$, then $d(Ty_n, Ty) \to 0 \; as \; n \to \infty$., therefore T is continuous.

3.2 Method of solution

Let the solution of equation (1) and equation (2) be approximated by

$$y(x) = \sum_{m=0}^{M} a_m x^m = \emptyset(x)\boldsymbol{A} \quad (11)$$

where $\emptyset(x) = [1 \;\; x \;\; x^2 \;\; \cdots \;\; x^M]$, $\boldsymbol{A} = [a_0 \;\; a_1 \cdots a_M]^T$

Substituting equation (11) into equation (8) gives

$$\emptyset(x)\boldsymbol{A} = W(x) - \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(q(x)\emptyset(x)\boldsymbol{A}\right)ds - \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(\lambda_1\int_0^x k_1(x,t)\,F_1(\emptyset(t)\boldsymbol{A})dt + \lambda_2\int_0^1 k_2(x,t)\,F_2(\emptyset(t)\boldsymbol{A})dt\right)ds \quad (12)$$

Collocating at x_i in equation (12) gives

$$\emptyset(x_i)\boldsymbol{A} = W(x_i) - \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(q(x)\emptyset(x_i)\boldsymbol{A}\right)ds - \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(\lambda_1\int_0^x k_1(x,t)\,F_1(\emptyset(t)\boldsymbol{A})dt + \lambda_2\int_0^1 k_2(x,t)\,F_2(\emptyset(t)\boldsymbol{A})dt\right)ds \quad (13)$$

Factorizing the value of **A** from equation (13) gives

$$\left[\left(\emptyset(x_i)+\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\left(q(x)\emptyset(x_i)\boldsymbol{A}\right)ds+\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\begin{pmatrix}\lambda^1\int_0^x k^1(x,t)F^1(\emptyset(t)\boldsymbol{A})dt+\\ \lambda_2\int_0^1 k_2(x,t)\,F_2(\emptyset(t)\boldsymbol{A})dt\end{pmatrix}\right)ds\right]\boldsymbol{A}=W(x_i) \quad (14)$$

Equation (14) can be in the form

$$J(x_i)\boldsymbol{A}=W(x_i) \quad (15)$$

where

$$J(x_i)=\emptyset(x_i)+\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\left(q(x)\emptyset(x_i)\boldsymbol{A}\right)ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\left(\lambda_1\int_0^x k_1(x,t)\,F_1(\emptyset(t)\boldsymbol{A})dt+\lambda_2\int_0^1 k_2(x,t)\,F_2(\emptyset(t)\boldsymbol{A})dt\right)ds$$

We solve the system of equations (15) for the unknown values and substitute the results into the approximate solution to obtain the numerical result.

3.3 Convergence of the method

Theorem 4. **(Convergence of method)** Let (X,d) be a metric space and $T\colon X\to X$ be a continuous mapping and $y_N(x), y_{N-1}(x)\varepsilon X$ be approximate solutions of equation (7). Let $\Delta_N(x)=|y_N(x)-y_{N-1}(x)|$. $If\ \ lim_{N\to 0}(\Delta_N(x))\to 0$, then the method converges to an exact solution.

Proof.

Let $y_1(x)$ and $y_2(x)$ be approximated by $y_N(x)=\sum_{n=0}^{M}a_nx^n=\emptyset(x)\boldsymbol{A}$ and $y_{N-1}(x)=\sum_{n=0}^{M}a_nx^n=\emptyset(x)\boldsymbol{B}$
Substituting the approximate solution into equation (8) gives

$$Ty_N(x)=W(x)-\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\left(q(x)\emptyset(x)\boldsymbol{A}\right)ds-\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\Big(\lambda_1\int_0^x k_1(x,t)\,F_1(\emptyset(x)\boldsymbol{A})dt+$$
$$\lambda_2\int_0^1 k_2(x,t)\,F_2(\emptyset(x)\boldsymbol{A})dt\Big)ds$$

Similarly

$$Ty_{N-1}(x)=W(x)-\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\left(q(x)\emptyset(x)\boldsymbol{B}\right)ds-\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\Big(\lambda_1\int_0^x k_1(x,t)\,F_1(\emptyset(x)\boldsymbol{B})dt+$$
$$\lambda_2\int_0^1 k_2(x,t)\,F_2(\emptyset(x)\boldsymbol{B})dt\Big)ds$$
$$|(Ty_n)(x)-(Ty)(x)|=\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\,q(x)\emptyset(x)|\boldsymbol{B}-\boldsymbol{A}|ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\left(\lambda^{1\int_0^x k^1(x,t)}F^1\emptyset(x)|\boldsymbol{B}-\boldsymbol{A}|dt+\lambda^{2\int_0^1 k_2(x,t)}F^2\emptyset(x)|\boldsymbol{B}-\boldsymbol{A}|dt\right)ds$$

Since $x\varepsilon[0,1]$ and $|B-A|\neq 0$, hence $lim_{N\to 0}\Delta_N(x)\to 0$.
Therefore, the method converges.

3.4 Numerical examples

In this section, we provide numerical illustrations to assess the applicability and accuracy of the method. Let the approximate and exact solutions be $y_n(x)$ and y(x) respectively. $Error_N=|y_n(x)-y(x)|$.

Example 1. (Hou *et al.*, 2021) Consider the first order V-FIDE

$$y'(x)+y(x)+\frac{1}{2}\int_0^x x\,y^2(t)dt-\frac{1}{4}\int_0^1 t\,y^3(t)dt=2x+x^2+\frac{1}{10}x^6-\frac{1}{32} \quad (16)$$

subject to initial condition

$$y(0)=0$$

The exact solution is $y(x)=x^2$

Solution 1.

The approximate solution of equation (16) at N=5 gives

$$y^5=-2.546574062734\times 10^{-15}-2.842170943040\times 10^{-14}x+1.000000000000x^2+1.818989403546\times 10^{-12}x^3+1.818989403546\times 10^{-12}x^4$$

Example 2. (Hou *et al.*, 2021) Consider the first order V-FIDE

$$y'(x) + y(x) - 2\int_0^x sin(x)\, y^2(t)dt = cos(x) + (1-x)sin(x) + cos(x)sin^2(x) \quad (17)$$

subject to initial condition

$$y(0) = 0$$

The exact solution is $y(x) = sin(x)$

Solution 2.

The approximate solution of equation (17) at N=6 gives

$$y_6 = -4.737052972000 \times 10^{-10} + 1.000001048642x - 0.1394239556e - 2x^2 - 0.155098519841x^3 - 0.36024075147e - 1x^4 + 0.54202200699e - 1x^5 - 0.10605020725e - 1x^6.$$

Example 3. (Hou *et al.*, 2021) Consider the first order V-FIDE

$$y'(x) + \int_0^x (y^2(t) - 2)\, dt = \frac{1}{5}x^5 \quad (18)$$

subject to initial condition

$$y(0) = 0$$

The exact solution is $y(x) = x^2$

Solution 3.

The approximate solution of equation (18) at N=4 gives

$$y_4 = -1.110223025000 \times 10^{-16}x + 1.000000000000x^2 - 3.552713679000 \times 10^{-15}x^4.$$

4. Results and Discussion

This section discusses the numerical results in the solved examples using the derived numerical approach. The result obtained for Example 1, as shown in Table 1, is that the approximate solution at $N = 5$ namely $y_5 = -2.546574062734 \times 10^{-15} - 2.842170943040 \times 10^{-14}x + 1.000000000000x^2 + 1.818989403546 \times 10^{-12}x^3 + 1.818989403546 \times 10^{-12}x^4$.

The numerical result converged to an exact solution, and this confirms that our method performed better than the method proposed by (Hou *et al.*, 2021).

Table 1. Exact and approximate solutions, and absolute error for Example 1

x	Exact	Our method$_{N=5}$	Error$_5$	(Hou *et al.*, 2021) Error$_{12}$
0.2	0.040000000000	0.040000000000	0.00	9.3974e-5
0.4	0.160000000000	0.160000000000	0.00	1.9813e-4
0.6	0.360000000000	0.360000000000	0.00	2.8891e-4
0.8	0.640000000000	0.640000000000	0.00	4.0786e-4
1.0	1.000000000000	1.000000000000	0.00	5.1647e-4

In numerical Example 2, as shown in Table 2, the approximate solution at $N = 6$ is $y_6 = -4.737052972000 \times 10^{-10} + 1.000001048642x - 0.1394239556e - 2x^2 - 0.155098519841x^3 - 0.36024075147e - 1x^4 + 0.54202200699e - 1x^5 - 0.10605020725e - 1x^6$. The numerical result is better than the result obtained by (Hou *et al.*, 2021) at $N = 12$.

Table 2. Exact and approximate solutions, and absolute error for Example 2

x	Exact	Our method$_{N=6}$	Error$_6$	(Hou *et al.*, 2021) Error$_{12}$
0.2	0.198669330800	0.198662679000	6.651800e-6	8.53966e-5
0.4	0.389418342300	0.389440411500	2.20692e-5	1.79180e-4
0.6	0.564642473400	0.564648677600	4.062042e-5	2.5968e-4
0.8	0.717356090900	0.717323556600	2.5674657-5	3.71839e-4
1.0	0.841470984800	0.841481394400	9.6104096e-5	5.01073e-4

The approximate solution obtained in Example 3 at $N = 4$ is $y_4 = -1.110223025000 \times 10^{-16}x + 1.000000000000x^2 - 3.552713679000 \times 10^{-15}x^4$. The numerical result converged to an exact solution, and this confirms that our method performed better than the method proposed by (Hou *et al.*, 2021), as shown in Table 3.

Table 3. Exact and approximate solutions, and absolute error for Example 3

x	Exact	Our method$_{N=4}$	Error$_4$	(Hou *et al.*, 2021) Error$_4$
0.2	0.040000000000	0.040000000000	0.00	4.6875e-17
0.4	0.160000000000	0.160000000000	0.00	2.77556e-17
0.6	0.360000000000	0.360000000000	0.00	1.11022e-16
0.8	0.640000000000	0.640000000000	0.00	2.22045e-16
1.0	1.000000000000	1.000000000000	0.00	1.11022e-15

5. Conclusions

For the numerical solution of the V-FIDEs, the collocation approach was investigated in this paper. This method is simple to apply, reliable, and efficient. For all computations in this work, Maple 18 was employed.

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