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Original Article

An extended weighted exponential distribution and its associated regression model

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Abstract

In this paper, we propose a new distribution for survival data. The distribution is a new generalization of the weighted exponential distribution in the Marshall-Olkin family. Statistical properties of the proposed distribution are studied and its sub-models are presented. The unknown parameters are estimated via a Bayesian approach. Simulation studies are conducted to assess the performance of the Bayesian estimator. Complete and censored data sets are analyzed to illustrate the potential of the proposed distribution. We also develop a regression model based on the proposed distribution. Finally, we compare the proposed model with other models.

Keywords: Bayesian approach, censored data, Marshall-Olkin family, weighted exponential distribution, regression model

1. Introduction

Survival data refers to information that tracks the time until the occurrence of a specific event of interest. This type of data is commonly encountered in various fields, including medical research, epidemiology, cancer studies, and engineering. Analysis of survival data provides valuable insights into the timing of events and contributes to informed decision making. However, few distributions have been proposed for survival data.

Several approaches are introduced to derive new distributions to obtain more flexible distributions. An interesting technique is adding parameters to original distributions. New distributions that are expanded from the original distributions provide more flexibility. Following this concept, Marshall and Olkin (1997) introduced a family by adding a new parameter to baseline distributions. This family is known as the Marshall-Olkin (MO) family.

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The cumulative distribution function (cdf) of the MO family is given by

$$F(y) = \frac{G(y)}{1-(1-\alpha)\big(1-G(y)\big)}, y \in R, \alpha > 0. \quad (1)$$

The corresponding probability density function (pdf) of the MO family is

$$f(y) = \frac{\alpha g(y)}{\big[1-(1-\alpha)\big(1-G(y)\big)\big]^2}, y \in R, \alpha > 0. \quad (2)$$

where $G(y)$ and $g(y)$ are the cdf and pdf of a baseline distribution, respectively.

Although many new distributions extended by the MO family are more flexible than their baseline distributions, most were proposed for neither censored data nor regression analysis (Algarni, 2021; Ikechukwu & Eghwerido, 2022; Javed, Nawaz, & Irfan, 2019; Mirmostafaee, Mahdizadeh, & Lemonte, 2017; Ristić & Kundu, 2015; Sabook & Pogány, 2016). The MO extended Weibull distribution (Ghitany, Al-Hussaini, & Al-Jarallah, 2005), and the MO extended Lomax distribution (Ghitany, Al-Awadhi, & Alkhalfan, 2007) were introduced for censored data, but they were not developed for regression models. In addition, the method of maximum likelihood was applied to estimate the parameters of these distributions. Gupta and Kundu (2009) proposed the weighted exponential (WE) distribution that is a generalization of the exponential distribution obtained by the method of Azzalini (1985). The pdf of the WE distribution having the shape and the scale parameters, $\theta > 0$ and $\lambda > 0$, is given by

$$g(y) = \frac{\theta+1}{\theta}\lambda e^{-\lambda y}\left(1 - e^{-\theta\lambda y}\right), \qquad y > 0. \tag{3}$$

The cdf of the WE distribution is given by

$$G(y) = 1 - \frac{1}{\theta}e^{-\lambda y}\left(\theta + 1 - e^{-\theta\lambda y}\right). \tag{4}$$

The pdf and cdf of the WE distribution are in explicit form. This distribution contains the exponential distribution, the gamma distribution, and the generalized exponential (GE) distribution (Gupta & Kundu, 1999) as special cases. Moreover, two real data sets were analyzed for the WE distribution and the WE distribution outperforms the GE, gamma, and Weibull distributions.

This paper proposes a new distribution for survival data. The distribution is an extension of the WE distribution by following the idea of Marshall and Olkin (1997), called the Marshall-Olkin weighted exponential (MOWE) distribution. Some properties and sub-models of the MOWE distribution are studied. Since maximum likelihood estimation is not suitable for small sample sizes, a Bayesian approach is applied in this paper. Complete and right censored data sets are analyzed by the MOWE distribution. Furthermore, we propose a regression model based on the MOWE distribution and we also show the potential of the proposed regression model by using a real data set. The MOWE distribution and the MOWE regression model provide satisfactory fits in the applications, and they outperform other competitive models. Hence, the proposed model can be used as an alternative in survival data analysis.

The rest of this paper is organized as follows. We introduce the MOWE distribution and its sub-models in Section 2. In Section 3, the unknown parameters of the proposed distribution are estimated by the Bayesian approach. A new regression model based on the proposed distribution is developed in Section 4. In Section 5, simulation studies are presented. Applications of the MOWE distribution and the MOWE regression model to survival data are shown in Section 6. Finally, discussion and conclusions are reported in Section 7.

2. The Marshall-Olkin Weighted Exponential Distribution

The new distribution, called the MOWE distribution, is introduced in this section. Furthermore, we define the MOWE distribution and present its special cases.

Theorem 1. If a random variable Y follows the MOWE distribution with parameters $\alpha,\ \theta$ and $\lambda > 0$, this is denoted by $Y \sim \text{MOWE}(\alpha, \theta, \lambda)$. The cdf and pdf of Y are, respectively.

$$F(y) = \frac{1 - \frac{1}{\theta}e^{-\lambda y}\left(\theta + 1 - e^{-\theta\lambda y}\right)}{1 - (1-\alpha)\left(\frac{1}{\theta}e^{-\lambda y}\left(\theta + 1 - e^{-\theta\lambda y}\right)\right)}, \tag{5}$$

$$f(y) = \frac{\alpha(\theta+1)\lambda e^{-\lambda y}\left(1 - e^{-\theta\lambda y}\right)}{\theta\left[1 - (1-\alpha)\left(\frac{1}{\theta}e^{-\lambda y}\left(\theta + 1 - e^{-\theta\lambda y}\right)\right)\right]^2}, \quad y > 0. \tag{6}$$

Pdf and cdf plots of the MOWE distribution with different parameter values are shown in Figure 1. The MOWE pdf is either a decreasing or a unimodal function.

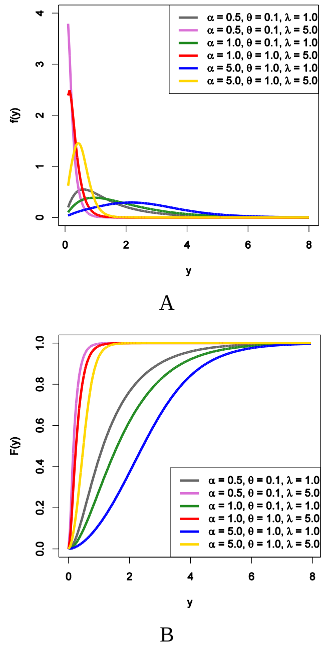


Figure 1. Pdf plot (A) and cdf plot (B) of the MOWE distribution with some parameter values

2.1 Sub-models of the MOWE distribution

Sub-models of the MOWE distribution are shown in Table 1.

2.2 Expansion

The pdf of MOWE distribution can be written in the form of a series expansion. It can be used to study properties of the MOWE distribution.

For $|z| < 1$ and $i > 0$, we can apply the series expansion

$$(1-z)^{-i} = \sum_{k=0}^{\infty}\binom{i+k-1}{k}z^k. \tag{7}$$

If $0 < \alpha < 1$, the pdf of the MOWE distribution is given by

Table 1. Sub-models of the MOWE distribution

Distribution	Parameter		
	α	θ	λ
The Marshall-Olkin exponential (MOE) distribution (Marshall & Olkin, 1997)	α	∞	λ
The Marshall-Olkin gamma (MOG) distribution (Ristić, Jose, & Ancy, 2007)	α	0	λ
The Marshall-Olkin generalized exponential distribution (Ristić & Kundu, 2015)	α	1	λ
The WE distribution (Gupta & Kundu, 2009)	1	θ	λ
The exponential distribution	1	∞	λ
The gamma distribution	1	0	λ
The GE distribution (Gupta & Kundu, 1999)	1	1	λ

$$f(y) = \alpha g(y) \sum_{k=0}^{\infty} (k+1)(1-\alpha)^k \sum_{j=0}^{k} \binom{k}{j} (-1)^j \, G(y)^j.$$

For $\alpha > 1$, we get

$$f(y) = \frac{g(y)}{\alpha\left[1 - \left(1 - \frac{1}{\alpha} G(y)\right)\right]^2}.$$

The other formula for the pdf of the MOWE distribution can be defined by

$$f(y) = \sum_{j=0}^{\infty} v_j w_{j+1}(y), \tag{8}$$

where

$$v_j = \begin{cases} \dfrac{\alpha(-1)^j}{j+1} \displaystyle\sum_{k=j}^{\infty} \binom{k}{j} (k+1)(1-\alpha)^k, & 0 < \alpha < 1 \\ \dfrac{1}{\alpha}\left(1 - \dfrac{1}{\alpha}\right)^j, & \alpha > 1 \end{cases}$$

and $w_{j+1}(y) = (j+1) g(y) G(y)^j$.

2.3 Survival function

The general form of the survival function or reliability function is given by $S(y) = 1 - F(y)$. Hence, the survival function of the MOWE distribution is

$$S(y) = \frac{\alpha e^{-\lambda y}\left(\theta + 1 - e^{-\theta\lambda y}\right)}{\theta\left[1 - (1-\alpha)\left(\frac{1}{\theta} e^{-\lambda y}(\theta + 1 - e^{-\theta\lambda y})\right)\right]}.$$

2.4 Hazard rate function

The hazard rate or failure rate is given by $h(y) = \frac{f(y)}{S(y)}$. Thus, the hazard rate of the MOWE distribution is

$$h(y) = \frac{(\theta+1)\lambda\left(1 - e^{-\theta\lambda y}\right)}{\left[1 - (1-\alpha)\left(\frac{1}{\theta} e^{-\lambda y}(\theta + 1 - e^{-\theta\lambda y})\right)\right]\left[\theta + 1 - e^{-\theta\lambda y}\right]}.$$

2.5 Reversed hazard rate function

The reversed hazard rate is expressed by $r(y) = \frac{f(y)}{F(y)}$. Therefore, the reversed hazard rate of the MOWE distribution is

$$r(y) = \frac{\alpha(\theta+1)\lambda e^{-\lambda y}\left(1 - e^{-\theta\lambda y}\right)}{\theta\left[1 - (1-\alpha)\left(\frac{1}{\theta} e^{-\lambda y}(\theta + 1 - e^{-\theta\lambda y})\right)\right]\left[1 - \frac{1}{\theta} e^{-\lambda y}(\theta + 1 - e^{-\theta\lambda y})\right]}.$$

2.6 Order statistic

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample of size n from the MOWE distribution and let $Y_{i:n}$, $1 \le i \le n$ denote the ***i*****-**th order statistics. Then, the pdf of $Y_{i:n}$ is written by

$$f_{i:n}(y) = \frac{n!}{(i-1)!\,(n-i)!} f(y)\left(F(y)\right)^{i-1}\left(1 - F(y)\right)^{n-i}. \tag{9}$$

Substituting the cdf and pdf of the MOWE distribution in Equation (5) and Equation (6) into Equation (9), the ***i*****-**th order statistic of the MOWE distribution is given by

$$f_{i:n}(y) = \frac{n!}{(i-1)!\,(n-i)!} \frac{\frac{\theta+1}{\theta}\alpha\,\lambda\, e^{-\lambda y}\left(1 - e^{-\theta\,\lambda\, y}\right)}{\left[1 - (1-\alpha)\left(\frac{1}{\theta} e^{-\lambda\, y}(\theta + 1 - e^{-\theta\,\lambda\, y})\right)\right]^2} \times \left(\frac{1 - \frac{1}{\theta} e^{-\lambda y}\left(\theta + 1 - e^{-\theta\lambda y}\right)}{1 - (1-\alpha)\left(\frac{1}{\theta} e^{-\lambda y}(\theta + 1 - e^{-\theta\lambda y})\right)}\right)^{i-1} \times \left(1 - \frac{1 - \frac{1}{\theta} e^{-\lambda y}\left(\theta + 1 - e^{-\theta\lambda y}\right)}{1 - (1-\alpha)\left(\frac{1}{\theta} e^{-\lambda y}(\theta + 1 - e^{-\theta\lambda y})\right)}\right)^{n-i}.$$

2.7 Quantile function

The quantile function is also known as the inverse of cdf. It provides a way to determine the value of random variable that corresponds to a specific probability. Moreover, it can be applied for skewness and kurtosis.

The ***p*****-**th quantile y_p of the MOWE distribution can be obtained by solving the equation of $F(y) = p$ to get

$$y_p = \frac{1}{\lambda}\log\left[\frac{\left(\theta + 1 - e^{-\theta\lambda y_p}\right)\left(1 - (1-p)\right)}{\theta(1-p)}\right],$$

where y_p can be used to generate the MOWE random variate.

Furthermore, the quantile function of the MO family can be written in term of the baseline quantile function as

$$y_p = Q_G\left(\frac{\alpha p}{1 - (1-\alpha)p}\right),$$

where Q_G is the baseline quantile function.

Hence, the quantile function of the MOWE distribution is written by

$$Q_p = Q_{WE}\left(\frac{\alpha p}{1-(1-\alpha)p}\right),$$

where Q_{WE} is the quantile function of the WE distribution.

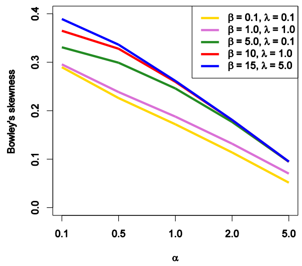
Quantiles can be used to calculate measures of skewness and kurtosis. The Bowley's skewness and the Moor's kurtosis of the MOWE distribution are given by

$$B_k = \frac{Q_{0.75}+Q_{0.25}-2Q_{0.5}}{Q_{0.75}-Q_{0.25}},$$

$$M_k = \frac{Q_{0.875}-Q_{0.625}-Q_{0.375}+Q_{0.125}}{Q_{0.75}-Q_{0.25}},$$

respectively.

Figure 2 shows Bowley's skewness plot and Moor's kurtosis plot of the MOWE distribution with some parameter values.



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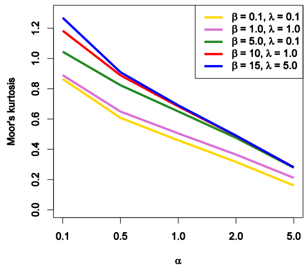

B

Figure 2. Bowley's skewness plot (A) and Moor's kurtosis plot (B) of the MOWE distribution with some parameter values

3. Parameter Estimation

In this paper, Bayesian estimators are considered to estimate the unknown parameters in the MOWE model. The Bayesian approach regards parameters as random variables represented by a prior distribution.

Suppose $Y_1, Y_2, \dots, Y_n$ be an independent and identically distributed random variable of size n and $y_1, y_2, \dots, y_n$ be the observations with the likelihood function $L(y|\Theta)$ where $\Theta = (\alpha, \theta, \lambda)$ is the vector of parameters. The independent prior density $\pi(\Theta)$ can be set as

$$\pi(\Theta) = \pi(\alpha)\pi(\theta)\pi(\lambda).$$

The joint posterior distribution is given by

$$\pi(\Theta|y) = \frac{L(y|\Theta)\pi(\Theta)}{\int_{\Theta} L(y|\Theta)\pi(\Theta)d(\Theta)}.$$

Since the denominator is normalization constant, the posterior distribution can be determined as

$$\pi(\Theta|y) \propto L(y|\Theta)\pi(\Theta). \quad (10)$$

3.1 Bayesian estimators based on complete data

Let $Y_1, Y_2, \dots, Y_n$ be a random variable of size n from the MOWE distribution with the vector of parameters Θ. Then the likelihood function of the observed sample is given by

$$L(y|\Theta) = \prod_{i=1}^{n} f(y_i;\Theta)$$
$$= \prod_{i=1}^{n} \frac{\alpha(\theta+1)\lambda e^{-\lambda y_i}\left(1-e^{-\theta\lambda y_i}\right)}{\theta\left[1-(1-\alpha)\left(\frac{1}{\theta}e^{-\lambda y_i}(\theta+1-e^{-\theta\lambda y_i})\right)\right]^2}.$$

In this paper, the gamma(a, b) distribution is used for non-informative prior distribution. According to Equation **(10)**, we get the joint posterior distribution for parameters as

$$\pi(\Theta|\, y) \propto \prod_{i=1}^{n} \frac{\alpha\,(\theta+1)\lambda\, e^{-\lambda\, y_i}\left(1-e^{-\theta\,\lambda\, y_i}\right)}{\theta\left[1-(1-\alpha)\left(\frac{1}{\theta}\, e^{-\lambda\, y_i}(\theta+1-e^{-\theta\,\lambda\, y_i})\right)\right]^2}$$
$$\times \frac{b_\alpha^{a_\alpha}}{\Gamma(a_\alpha)}\alpha^{a_\alpha-1}e^{-b_\alpha\alpha} \times \frac{b_\theta^{a_\theta}}{\Gamma(a_\theta)}\theta^{a_\theta-1}e^{-b_\theta\theta}$$
$$\times \frac{b_\lambda^{a_\lambda}}{\Gamma(a_\lambda)}\lambda^{a_\lambda-1}e^{-b_\lambda\lambda}.$$

The LaplaceDemon function maximizes the logarithm of the joint posterior density, and then

$$\log(\pi(\Theta|y)) \propto \log(L(y|\Theta)) + \log(\pi(\Theta)).$$

The joint posterior distribution for parameters of the MOWE distribution is obtained by

$$\log(\pi(\Theta|y)) \propto \sum_{i=1}^{n}\log(\alpha) - \sum_{i=1}^{n}\log(\theta) + \sum_{i=1}^{n}\log(\theta+1) + \sum_{i=1}^{n}\log(\lambda)$$
$$-\lambda\sum_{i=1}^{n} y_i + \sum_{i=1}^{n}\log\left(1-e^{-\theta\lambda y_i}\right)$$
$$-2\sum_{i=1}^{n}\log\left(1-(1-\alpha)\left(\frac{1}{\theta}e^{-\lambda y_i}(\theta+1-e^{-\theta\lambda y_i})\right)\right)$$
$$+\log\left(\frac{b_\alpha^{a_\alpha}}{\Gamma(a_\alpha)}\alpha^{a_\alpha-1}e^{-b_\alpha\alpha}\right) + \log\left(\frac{b_\theta^{a_\theta}}{\Gamma(a_\theta)}\theta^{a_\theta-1}e^{-b_\theta\theta}\right)$$
$$+\log\left(\frac{b_\lambda^{a_\lambda}}{\Gamma(a_\lambda)}\lambda^{a_\lambda-1}e^{-b_\lambda\lambda}\right).$$

3.2 Bayesian estimators based on censored data

Considering an observation (y_i, δ_i) where y_i is the failure time and δ_i is the censoring indicator ($\delta_i = 0$ if ***i***-th observation is censored or $\delta_i = 1$ if ***i***-th observation is recorded). The likelihood function of the MOWE distribution under censored sample is given by

$$L(y|\Theta) = \prod_{i=1}^{n} f(y_i;\Theta)^{\delta_i} S(y_i;\Theta)^{1-\delta_i}$$
$$= \prod_{i=1}^{n}\left[\frac{\alpha(\theta+1)\lambda e^{-\lambda y_i}\left(1-e^{-\theta\lambda y_i}\right)}{\theta\left[1-(1-\alpha)\left(\frac{1}{\theta}e^{-\lambda y_i}(\theta+1-e^{-\theta\lambda y_i})\right)\right]^2}\right]^{\delta_i}$$

$$\times\left[\frac{\alpha e^{-\lambda y_i}\left(\theta+1-e^{-\theta\lambda y_i}\right)}{\theta\left[1-(1-\alpha)\left(\frac{1}{\theta}e^{-\lambda y_i}(\theta+1-e^{-\theta\lambda y_i})\right)\right]}\right]^{1-\delta_i}.$$

The joint posterior density for parameters is obtained by

$$\pi(\Theta|y)\propto\prod_{i=1}^{n}\left[\frac{\alpha(\theta+1)\lambda e^{-\lambda y_i}\left(1-e^{-\theta\lambda y_i}\right)}{\theta\left[1-(1-\alpha)\left(\frac{1}{\theta}e^{-\lambda y_i}(\theta+1-e^{-\theta\lambda y_i})\right)\right]^2}\right]^{\delta_i}$$
$$\times\left[\frac{\alpha e^{-\lambda y_i}\left(\theta+1-e^{-\theta\lambda y_i}\right)}{\theta\left[1-(1-\alpha)\left(\frac{1}{\theta}e^{-\lambda y_i}(\theta+1-e^{-\theta\lambda y_i})\right)\right]}\right]^{1-\delta_i}$$
$$\times\frac{b_\alpha^{a_\alpha}}{\Gamma(a_\alpha)}\alpha^{a_\alpha-1}e^{-b_\alpha\alpha}\times\frac{b_\theta^{a_\theta}}{\Gamma(a_\theta)}\theta^{a_\theta-1}e^{-b_\theta\theta}$$
$$\times\frac{b_\lambda^{a_\lambda}}{\Gamma(a_\lambda)}\lambda^{a_\lambda-1}e^{-b_\lambda\lambda}.$$

Based on the Laplace's Demon, the logarithm of the unnormalized joint posterior distribution is given by

$$\log\left(\pi(\Theta|y)\right)\propto\sum_{i=1}^{n}\log(\alpha)-\sum_{i=1}^{n}\log(\theta)+\sum_{i=1}^{n}\delta_i\log(\theta+1)$$
$$+\sum_{i=1}^{n}\delta_i\log(\lambda)\ -\lambda\sum_{i=1}^{n}y_i+\sum_{i=1}^{n}\delta_i\log\left(1-e^{-\theta\lambda y_i}\right)$$
$$+\sum_{i=1}^{n}(1-\delta_i)\log\left(\theta+1-e^{-\theta\lambda y_i}\right)$$
$$-\sum_{i=1}^{n}(1+\delta_i)\log\left(1-(1-\alpha)\left(\frac{1}{\theta}e^{-\lambda y_i}\left(\theta+1-e^{-\theta\lambda y_i}\right)\right)\right)$$
$$+\log\left(\frac{b_\alpha^{a_\alpha}}{\Gamma(a_\alpha)}\alpha^{a_\alpha-1}e^{-b_\alpha\alpha}\right)+\log\left(\frac{b_\theta^{a_\theta}}{\Gamma(a_\theta)}\theta^{a_\theta-1}e^{-b_\theta\theta}\right)$$
$$+\log\left(\frac{b_\lambda^{a_\lambda}}{\Gamma(a_\lambda)}\lambda^{a_\lambda-1}e^{-b_\lambda\lambda}\right).$$

The joint posterior distribution is not in closed form, but the Markov Chain Monte Carlo (MCMC) methods can be applied to obtain the posterior distribution. In this paper, the Metropolis-Hastings algorithm within Gibbs technique, with 10,000 iterations and a burn-in of 5,000 samples in the LaplaceDemon package (Statisticat, 2016) of the R programming language (R Core Team, 2023), is employed.

4. The Marshall-Olkin Weighted Exponential Regression Model

Let $\boldsymbol{x_i}=\left(1,x_{i1},x_{i2},\dots,x_{ip}\right)^T$ be the vector of covariates. The parameter $\boldsymbol{\lambda_i}$ is linked to the covariates by the logarithmic link function $\log(\boldsymbol{\lambda_i})=\boldsymbol{x_i}^T\boldsymbol{\beta}$, where $i=1,2,\dots,n$ and $\boldsymbol{\beta}=(\beta_0,\beta_1,\dots,\beta_p)$ is the vector of regression coefficients.

Thus, the pdf of the MOWE regression can be defined by

$$f(y)=\frac{\alpha(\theta+1)e^{x_i^T\beta}e^{-e^{x_i^T\beta}y}\left(1-e^{-\theta e^{x_i^T\beta}y}\right)}{\theta\left[1-(1-\alpha)\left(\frac{1}{\theta}e^{-e^{x_i^T\beta}y}\left(\theta+1-e^{-\theta e^{x_i^T\beta}y}\right)\right)\right]^2}.$$

The corresponding survival function is given by

$$S(y)=\frac{\alpha e^{-e^{x_i^T\beta}y}\left(\theta+1-e^{-\theta e^{x_i^T\beta}y}\right)}{\theta\left[1-(1-\alpha)\left(\frac{1}{\theta}e^{-e^{x_i^T\beta}y}\left(\theta+1-e^{-\theta e^{x_i^T\beta}y}\right)\right)\right]}.$$

In this paper, the prior distributions for all unknown parameters considered are

$\alpha\sim$ gamma(0.001,0.001),
$\theta\sim$ gamma(0.001,0.001),
$\boldsymbol{\beta}\sim$ N(0, 10000).

We apply the Metropolis-Hastings within Gibbs technique with 10,000 iterations and a burn-in of 5,000 samples the LaplaceDemon package (Statisticat, 2016) in the R programming language (R Core Team, 2023) for the Bayesian estimates.

5. Simulation Study

In this section, Monte Carlo simulations are conducted to assess the performance of the Bayesian estimators. Two simulation studies are presented. The first simulation evaluates the Bayesian estimators for complete samples and the second one for censored samples. The inversion method is used to generate samples and the survival package (Therneau, 2023) in the R programming language (R Core Team, 2023) is used to generate censored samples. The simulations are carried out 1,000 times with $\alpha=0.5$, $\beta=0.5$ and $\lambda=0.5$ for the different sample sizes $n=20, 50, 100, 200$. We calculate the averages of the Bayesian estimates. The performance measures are based on root mean square error (RMSE) and average bias, defined by

$$\text{RMSE}=\sqrt{\frac{\sum_{i=1}^{1000}(\widehat{\Theta}_i-\Theta)^2}{1000}},$$
and $$\text{Average bias}=\frac{\sum_{i=1}^{1000}(\widehat{\Theta}_i-\Theta)}{1000}.$$

Tables 2 and 3 display simulation results from the MOWE model based on complete samples and right censored samples, respectively. The averages of the Bayesian estimates are close to the true values in both simulation studies. The average biases approach zero as the sample size increase and the RMSEs decreases to zero with increasing sample size.

6. Applications

The MOWE distribution and the MOWE regression model are tested for performance in this section. The MOWE distribution is fitted to censored and non-censored real data sets and the distribution is compared with the exponential, gamma, WE, MOE, and MOG distributions. In the third application, the exponential, gamma, WE, MOE, MOG, and MOWE regression models are compared by using a censored data set. In this paper, the LaplaceDemon function (Statisticat, 2016) in the R programming language (R Core Team, 2023) is used for all models. The model selection is carried out by the log-likelihood (log L) value, the Akaike Information Criterion (AIC) (Akaike, 1974), and the Deviance Information Criterion (DIC) (Spiegelhalter, Best, Carlin, & Van Der Linde, 2002). The largest log L, smallest AIC, and smallest DIC values indicate the best model.

The deviance is given by

$$D(\Theta) = -2\log L(\Theta|y) + c,$$

where c is a constant that cancels out on comparing models.

The DIC can be defined by

$$D(\widehat{\Theta}) + 2p_D,$$

where $D(\widehat{\Theta})$ is the deviance estimated at the posterior mean of $\widehat{\Theta}$ and p_D is the effective number of parameters.

The deviance can be used to calculate the AIC by

$$\text{AIC} = -2\log L(\widehat{\Theta}|y) + 2p,$$

where p is the number of model parameters.

6.1 Application 1 non-censored data

The first data set describes 44 survival times of patients suffering from head and neck cancer diseases. Patients were treated by using a combination of radiotherapy and chemotherapy (Shanker, Fesshaye, & Selvaraj, 2015). The observations are as follows.

Table 5 shows the posterior means, log L, AIC, and DIC values for survival times of patients suffering from head and neck cancer. From the results in Table 3, we can see that the MOWE provides the largest log L, the smallest AIC and the smallest DIC values.

6.2 Application 2 censored data

The second data set presents censored survival times for head and neck cancer patients treated by chemotherapy plus radiation (Efron, 1988). The observations are as in Table 6.

Table 7 displays the posterior means, log L, AIC, and DIC values of all distributions fitted to censored survival times of head and neck cancer. The table shows that the MOWE distribution again provides the largest log L, the smallest AIC and the smallest DIC values.

Figure 3 shows the Kaplan-Meier survival curve (empirical survival function plot) and estimated survival function plots for exponential, gamma, WE, MOE, MOG, and MOWE models. The figure shows that the MOWE model is the best one among the competing models because it is closest to the Kaplan-Meier survival curve.

6.3 Application 3 regression model with censored data

The NKI breast cancer clinical data (Putter & Putter, 2011; Van De Vijver *et al.*, 2002; Van Houwelingen *et al.*, 2006; Van’t Veer *et al.*, 2002) is considered to compare the exponential, gamma, WE, MOE, MOG, and MOWE regression

Table 2. RMSE and average bias from the simulated MOWE model based on complete samples.

n	α		θ		λ	
	RMSE	Average bias	RMSE	Average bias	RMSE	Average bias
20	2.308	1.003	2.988	1.330	0.329	0.051
50	0.811	0.355	1.822	0.870	0.190	0.035
100	0.471	0.199	0.858	0.373	0.140	0.050
200	0.275	0.083	0.325	0.093	0.108	0.048

Table 3. RMSE and average bias from the simulated MOWE model based on censored samples.

n	α		θ		λ	
	RMSE	Average bias	RMSE	Average bias	RMSE	Average bias
20	0.365	–0.262	3.588	–2.961	0.320	–0.307
50	0.354	–0.260	1.166	–0.524	0.284	–0.259
100	0.331	–0.251	0.799	–0.402	0.235	–0.220
200	0.296	–0.212	0.678	–0.369	0.194	–0.170

Table 4. Non-censored survival times of patients suffering from head and neck cancer diseases

12.20	23.56	23.74	25.87	31.98	37	41.35	47.38	55.46	58.36
63.47	68.46	78.26	74.47	81.43	84	92	94	110	112
119	127	130	133	140	146	155	159	173	179
194	195	209	249	281	319	339	432	469	519
633	725	817	1776						

Table 5. Posterior means, log L, AIC, and DIC values for fitted distributions to survival time data

Distribution	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$	log L	AIC	DIC
Exponential	-	-	0.0046	–282.186	566.371	564.540
Gamma	-	1.0100	0.0045	–283.270	570.540	568.207
WE	-	28.4024	0.0004	–281.608	567.215	564.365
MOE	0.4866	-	0.0029	–281.657	567.314	564.174
MOG	0.1700	1.6951	0.0034	–279.041	564.082	560.030
MOWE	0.0521	0.0005	0.0278	–278.689	563.379	557.812

Table 6. Censored data survival times for head and neck cancer patients treated

37	84	92	94	110	112	119	127	130	133
140	146	155	159	169+	173	179	194	195	209
249	281	319	339	432	469	519	528+	547+	613+
633	725	759+	817	1092+	1245+	1331+	1557	1642+	1771+
1776	1897+	2023+	2146+	2297+					

Table 7. Posterior means, log *L*, AIC, and DIC values for distributions fitted to censored survival times data

Distribution	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$	log *L*	AIC	DIC
Exponential	-	-	0.0013	−243.908	489.815	488.962
Gamma	-	0.8286	0.0009	−243.201	490.403	489.898
WE	-	40.8633	0.0011	−242.463	488.926	485.722
MOE	0.2176	-	0.0004	−238.890	481.780	478.533
MOG	0.1579	1.2416	0.0005	−239.380	484.760	480.343
MOWE	0.2266	36.1203	0.0005	−237.632	481.264	477.155

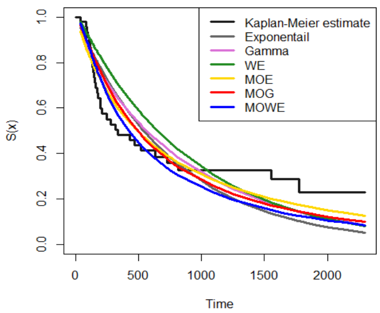

Figure 3. Kaplan-Meier curve and estimated survival function plots

models. The data frame consists of 295 breast cancer patients in the Dutch Cancer Institute (NKI), Amsterdam. The variables used are as follows:

- *tyears*, time (year) until death or last follow-up
- *d*, survival status (1 = death; 0 = censored)
- *diameter*, the primary tumor's diameter
- *age*, the patient's age

Table 8 shows the log *L*, AIC, and DIC values of all regression models fitted to NKI data. The table indicates that the MOWE regression model is suitable for the data set because the model provides the highest log *L*, the lowest AIC and the lowest DIC values.

Table 9 displays posterior means, the lower bound (LB) and the upper bound (UB) of the 95% credible interval for parameters of the MOWE regression model. We can conclude that all parameters are significant at the level of 0.05.

7. Discussion and Conclusions

This article introduced a new generalized form of the WE distribution, called the Marshall-Olkin weighted exponential (MOWE) distribution. The distribution is obtained with the MO family transform based on the WE distribution. This model was proposed for survival data. Special cases of the proposed distribution were presented. Bayesian estimators of parameters were derived. The proposed distribution was compared with the exponential, gamma, WE, MOE, and MOG distributions by using complete and censored data. Furthermore, a new regression model based on the MOWE distribution was constructed. We illustrated the usefulness of the MOWE regression model with a real data set. The proposed model provided better fits than other competitive models. However, the pdf of MOWE distribution has a quite complicated formula and many parameters. In this paper, the Bayesian approach was applied for identifying the parameters, but it takes a long time to process. In a future study, other parameter estimation methods will be considered.

Table 8. log *L*, AIC, and DIC values for regression models fitted to NKI data

Model	log *L*	AIC	DIC
Exponential	−340.203	686.406	682.076
Gamma	−338.463	684.927	680.438
WE	−334.491	676.982	671.210
MOE	−338.914	685.829	679.721
MOG	−336.110	682.219	675.589
MOWE	−331.426	672.852	665.913

Table 9. Posterior means and the 95% credible intervals of parameters for the MOWE regression model fit to NKI data

Parameter	Posterior mean	LB	UB
$\widehat{\beta_0}$	−3.666	−4.941	−2.402
$\widehat{\beta_1}$	0.041	0.026	0.057
$\widehat{\beta_2}$	−0.059	−0.070	−0.048
$\hat{\alpha}$	0.115	0.027	0.346
$\hat{\theta}$	92.049	29.218	165.217

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