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Original Article

Sequences space ℓ_p^N of neutrosophic real numbers

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Abstract

The article reveals concept of $\ell_p^N(X), 1 \leq p < \infty$, the p – absolutely summable neutrosophic valued sequence space. We have proved that it is a neutrosophic normed linear space valued sequence space. We have studied its different algebraic and topological properties. We have also established some inclusion results. The main aim is to introduce the notion of neutrosophic normed linear space $\ell_p^N(X), 1 \leq p < \infty$, and to show that it is a complete neutrosophic sequence space.

Keywords: neutrosophic normed linear space, convergence, solid space, monotone, symmetricity

1. Introduction

In today's world we come across with many uncertainties which cannot always be dealt with the help of classical methods. To overcome such uncertainties, Zadeh (1965) introduced the concept of fuzzy sets. Regardless of many applications of it, fuzzy set cannot explain the indeterminacy states as it provides a truth value only. Kamthan and Gupta (1980) investigated sequence spaces and series. The notion of fuzzy metrices was introduced by Kelava and Seikkala (1984). Then Atanassov (1986) revealed intuitionistic fuzzy sets theory considering non - membership value along with membership one. Felbin (1992) investigated finite dimensional fuzzy normed linear spaces. Thereafter, concept of neutrosophic set was introduced by Smarandache (1998, 1999). Here membership, non - membership and indeterminacy are defined as independent of each other in neutrosophic set theory. Intuitionistic fuzzy set theory has a role to play in all areas of research where fuzzy set theory has been applied. Park (2004) defined metric space in intuitionistic fuzzy setting. Smarandache (2005) further investigated neutrosophic sets. Researchers successfully applied the notion of fuzzy and intuitionistic fuzzy sets theory in studying several types of sequence spaces viz. (Das 2014a, 2014b; Karakus, Demirci, & Duman, 2008; Karakaya, Simsek, Erturk, & Gursoy, 2014; Komisaski, 2008; Kumar & Kumar, 2009; Saadati, 2009; Tripathy, Baruah, & Gungor, 2012; Tripathy & Dutta, 2014, 2015; Tripathy & Sarma, 2008. Further Das (2017) introduced normed linear space sequence space in fuzzy setting. Bera and Mahapatra (2017, 2018) studied the neutrosophic soft linear spaces and their normed spaces. Tripathy and Das (2019) investigated a class of fuzzy number sequences bv_p^F. Muralikrishna and Kumar (2019) investigated neutrosophic approach to normed linear space. Kirişci and Şimşek (2020) investigated statistical convergence in neutrosophic normed space (NNS). Omer (2021a, 2021b, 2022) investigated different types of convergence in neutrosophic normed spaces. Studies on different types of statistical convergence in neutrosophic normed spaces have been carried out by Gonul (2022, 2023a and 2023b) and Khan *et al.* (2023). In section 2, we mention some definitions and results relevant to the study.

Main contribution: The main contributions are as follows:

- We have defined neutrosophic normed linear space valued sequence space. We have also defined addition and scalar multiplication, monotonicity, symmetric, convergence, solidness, completeness, p – absolutely summable sequence etc. of this space.

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- We have established that $p-$ absolutely summable sequence space is neutrosophic normed linear space valued sequence space.
- We have obtained completeness property of neutrosophic normed linear space valued sequence space.
- With suitable counterexample, we have established that neutrosophic normed linear space valued sequence space is not convergence free
- We have established symmetricity and inclusion property of neutrosophic normed linear space valued sequence space.

2. Definitions and Preliminaries

Here we procured different well-known definitions and results, which are used in getting our results. For Definitions 2.1 to 2.8, one may refer to Das (2017); and for Definition 2.9, one may refer to Smarandache (1998).

Definition 2.1. Let the collection of all bounded intervals $[a_1, a_2]$ on $\mathbb{R}$ be C. Let $X, Y \in C$.
Take

$$U = [x_1, x_2] \text{ and } V = [y_1, y_2]. \text{ For } x_1 \leq y_1 \text{ and } x_2 \leq y_2, \text{ let } U \leq V \text{ and}$$
$$d(U, V) = \max(|x_1 - y_1|, |x_2 - y_2|).$$

Then, $\leq$ is a partial order in C and (C, d) is a complete metric space.

Definition 2.2. The fuzzy real numbers are denoted by $\mathbb{R}^F = \{(x, \mu_{\mathbb{R}}(x)) : x \in \mathbb{R}\}$ where $\mu_{\mathbb{R}} : \mathbb{R} \in [0,1]\}$.

Definition 2.3. Let $[X]^\alpha$ be denote α-level set of fuzzy real numbers X where $0 < \alpha \leq 1$ and $[F]^\alpha = \{t \in \mathbb{R} : F(t) \geq \alpha\}$.
We get closure of strong 0-cut for $\alpha = 0$.
Collection $\{t \in \mathbb{R} : X(t) > \alpha\}$ denotes strong α-cut, where $0 \leq \alpha \leq 1$.
Take $X, Y \in \mathbb{R}(I)$ and consider $\leq$, a partial ordering defined by

$$X \leq Y \text{ if and only if } a_1^\alpha \leq a_2^\alpha \text{ and } b_1^\alpha \leq b_2^\alpha, \forall \alpha \in (0,1],$$

where $[X]^\alpha = [a_1^\alpha, b_1^\alpha]$ and $[Y]^\alpha = [a_2^\alpha, b_2^\alpha]$.

Definition 2.4. Consider $X, Y \in \mathbb{R}(I)$ and let their α-level sets be $[X]^\alpha = [a_1^\alpha, b_1^\alpha]$, $[Y]^\alpha = [a_2^\alpha, b_2^\alpha], \alpha \in [0,1]$. We define arithmetic operations on ℝ(I) with $\alpha-$ level sets as below:

$$[X \oplus Y]^\alpha = [a_1^\alpha + a_2^\alpha, b_1^\alpha + b_2^\alpha],$$
$$[X \ominus Y]^\alpha = [a_1^\alpha - b_2^\alpha, b_1^\alpha - a_2^\alpha],$$
$$[X \otimes Y]^\alpha = [\min_{i,j\in\{1,2\}} a_i^\alpha b_j^\alpha, \min_{i,j\in\{1,2\}} a_i^\alpha b_j^\alpha]$$
$$\text{and } [\bar{1} \div Y]^\alpha = \left[\frac{1}{b_2^\alpha}, \frac{1}{a_2^\alpha}\right], 0 \notin Y$$

Definition 2.5. Let F be a fuzzy real number. Then its absolute value $|F|$ of $X \in \mathbb{R}(I)$ is given by

$$|F|(t) = \begin{cases} \max(F(t), F(-t)), & \text{for } t \geq 0, \\ 0, & \text{for } t < 0. \end{cases}$$

Definition 2.6. A fuzzy real number F is said to be non-negative whenever $F(t)$ takes zero $\forall\, t < 0$ and we denote such collection by $\mathbb{R}^*(I)$.

Definition 2.7. Consider V as a vector space, and let $\omega(V)$ denote the collection of sequences in V. Then with respect to pointwise addition and scalar multiplication, $\omega(V)$ is a vector space. If $\Gamma(V)$ is a subspace of $\omega(V)$, then it is said to be vector valued as well as fuzzy normed linear space-valued sequence space on fuzzy normed linear space $(X, ||\cdot||)$.

Definition 2.8. Let X be a vector space over R. Assume that the mappings L, R : [0, 1] × [0, 1] → $[0,1]$ *are symmetric and non* − decreasing in both arguments and that L(0, 0) = 0 and R(1, 1) = 1. Let $||\cdot|| : X \to$ F*(R). The quadruple (X, $||\cdot||$, L, R) is called a fuzzy normed space with the fuzzy number $||\cdot||$, if the following conditions are satisfied:

(i) If $x \neq 0$, then $\inf||x||_\alpha > 0$ whenever $0 < x < 1$.
(ii) $||x|| = \bar{0}$ if and only if x = 0.
(iii) $||rx|| = |r|||x||$ for x ∈ X and r ∈ R.
(iv) *for all* $x, y \in X$,
$(a)||x+y||(s+t) \geq L\left(||x||(s), ||y||(t)\right)$ whenever $s \leq ||x||_1$, $t \leq ||y||$, $s + t \leq ||x+y||_1$.
$(b)||x+y||(s+t) \leq L\left(||x||(s), ||y||(t)\right)$ whenever $s \geq ||x||_1$, $t \geq ||y||$, $s + t \geq ||x+y||_1$.

Definition 2.9. The neutrosophic real numbers are denoted by $\mathbb{R}^N = \{(x, T_{\mathbb{R}}(x), F_{\mathbb{R}}(x), I_{\mathbb{R}}(x)) : x \in \mathbb{R}\}$ where $T_{\mathbb{R}} : \mathbb{R} \to [0,1]$, $F_{\mathbb{R}} : \mathbb{R} \to [0,1]$, $I_{\mathbb{R}} : \mathbb{R} \to [0,1]$.

3. Neutrosophic Normal Linear Space

Now we introduce the notions of normed linear space in neutrosophic set of real numbers.

Definition 3.1. We denote linear space over $\mathbb{R}$ by X. Consider mapping $||\cdot||: X \rightarrow \mathbb{R}^*(I)$. We also consider symmetric mappings $L,\ M: [0,1] \times [0,1] \rightarrow [0,1]$ which is non-decreasing in both arguments and satisfies $L(0,0) = 0,\ M(1,1) = 1$. Define $[||x||]^\alpha = \max\{|x|_1^\alpha,\ |x|_2^\alpha\}$, for $x \in X$, $x^\alpha = [x_1^\alpha,\ x_2^\alpha], 0 < \alpha \leq 1$ and suppose $\forall x \in X,\ x \neq 0, \exists\ \alpha_0 \in (0,1]$ independent of x such that $\forall\ \alpha \leq \alpha_0$,

(a) $|x|_2^\alpha < \infty$,

(b) $inf_{\alpha \in (0,1]}[|x|_1^\alpha\} > 0$.

Then $(X, ||\cdot||,\ L,\ M)$ is said to be a neutrosophic normed linear space. Here $||\cdot||$ denotes neutrosophic norm, provided

(i) $||x^T|| = ||x^I|| = ||x^F|| = \bar{0}$ if and only if $x = \bar{\theta}$, the null element of X.

(ii) $||rx^T|| = |r|||x^T||,\ ||rx^I|| = |r|||x^I||, ||rx^F|| = |r|||x^F||, x \in X, r \in \mathbb{R}$.

(iii) $for\ all\ x, y \in X,$

$$(a)||x^T + y^T||(s+t) \geq L\left(||x^T||(s), ||y^T||(t)\right),$$
$$||x^I + y^I||(s+t) \geq L\left(||x^I||(s), ||y^I||(t)\right)$$
$$||x^F + y^F||(s+t) \geq L\left(||x^F||(s), ||y^F||(t)\right),$$
$$\text{whenever } s \leq |||x^T|||_1^1, |||x^I|||_1^1, |||x^F|||_1^1$$
$$t \leq |||y^T|||_1^1, |||y^I|||_1^1, |||y^F|||_1^1$$
$$s + t \leq |||x^T + y^T|||_1^1, |||x^I + y^I|||_1^1, |||x^F + y^F|||_1^1.$$
$$\text{(b) } ||x^I + y^I||(s+t) \geq M(||x^I||)(s), ||y^I||(t))$$
$$||x^T + y^T||(s+t) \geq M(||x^T||)(s), ||y^T||(t))$$
$$||x^F + y^F||(s+t) \geq M(||x^F||)(s), ||y^F||(t))$$
$$\text{whenever } s \geq |||x^T|||_1^1, |||x^I|||_1^1, |||x^F|||_1^1$$
$$t \geq |||y^T|||_1^1, |||y^I|||_1^1, |||y^F|||_1^1$$
$$s + t \geq |||x^T + y^T|||_1^1, |||x^I + y^I|||_1^1, |||x^F + y^F|||_1^1$$

In the sequel we take $L(x,y) = \min(x,y)$ and $M(x,y) = \max(x,y)$ for $x, y \in [0,1]$ and we here consider the space $(X, ||\cdot||, L, M)$, for short denoted by $(X, ||\cdot||)$ or simply by X.

Here $||x^T||, ||x^I||$ and $||x^F||$ denote symbolically the norms of truthness, indetermancy and falseness, respectively.

Definition 3.2. Consider X that is a vector space. Let $\omega(X)$ be the set of all sequences of it. Then with respect to pointwise addition and scalar multiplication, $\omega(X)$ is a vector space. Consider a subspace $\Gamma(X)$ of $\omega(X)$. Then $\Gamma(X)$ is said to be vector valued sequence space and it is called neutrosophic normed linear space valued sequence space when $(X,\ ||\cdot|)$ is a neutrosophic normed linear space.

Definition 3.3. Consider E^N(X) that is a neutrosophic normed linear space valued sequence space. Then it is called normal (or solid) if $(y_k) \in E^N$(X), as and when $||y_k^T|| \leq ||x_k^T||, ||y_k^I|| \geq\ ||x_k^I||, ||y_k^F|| \geq\ ||x_k^F||$, for all k $\in \mathbb{N}$ and $(x_k) \in E^N$(X).

Definition 3.4. The $E^N(X)$, a neutrosophic normed linear space valued sequence space, is called monotone when it takes canonical pre-images of all its step sets.

In Kamthan and Gupta (1980), for the sequences of real or complex numbers, a sequence space is solid implies it is monotone. This also holds for the sequences of fuzzy real numbers. In view of these, we have the following remark.

Remark 3.5. A solid neutrosophic normed linear space valued sequence space implies that it is also monotone.

Definition 3.6. A neutrosophic normed linear space valued sequence space $E^N(X)$ is termed symmetric when $(x_{\pi(n)}) \in E^N(X)$, for$(x_k) \in E^N(X)$. Here π indicates a permutation of $\mathbb{N}$.

Definition 3.7. A neutrosophic normed linear space valued sequence space $E^N(X)$ is called convergence free when $(y_k) \in E^N(X)$, for$(x_k) \in E^N(X)$ and $y_k = \bar{0}$
whenever $x_k = \bar{0}$.

Definition 3.8. A neutrosophic normed linear space $(X, ||\cdot||)$ is called complete whenever each of its Cauchy sequences converges to some point of it.

Definition 3.9. The neutrosophic normed linear space valued sequence space $\ell_p^N(X)$ can be defined with the help of neutrosophic norm as given below:

$$\ell_p^N(X) = \{x = (x_k) \in\ w^N(X): \textstyle\sum_{k=1}^{\infty} ||x_k||^p \leq \lambda, for\ some\ \lambda \in \mathbb{R}^*(I)\}.$$

Definition 3.10. The collection of all p- absolutely summable sequences in $(X, ||\cdot||)$ for a sequence $x = (x_k) \in l_p^N(X), 1 \le p < \infty$, can be defined as below:

$$||x|| = \{\sum_{k=1}^{\infty}(||x_k^T||^p + ||x_k^I||^p + ||x_p^F||^p)\}^{1/p}$$

Here $||x||$ indicates a norm on X.
Here $w^N(X)$, $\ell_p^N(X)$ and $c^N(X)$ indicate spaces of all, p-absolutely summable, and convergent sequences of X, respectively.

4. Prime Findings

Theorem 4.1. The collection of p-absolutely summable sequences $\ell_p^N(X)$ is neutrosophic normed linear space valued sequence space.

Proof. Consider $(X, ||\cdot||)$ as a neutrosophic normed linear space valued sequence space and $x = (x_k), y = (y_k) \in \ell_p^N(X)$. We have for

$$k \in \mathbb{N}, ||x_k + y_k||^p \le 2^p \max\{(||x_k^T||^p + ||x_k^I||^p + ||x_k^F||^p), (||y_k^T||^p + ||y_k^I||^p + ||y_k^F||^p)\}$$
$$\le 2^p\{||x_k^T||^p + ||x_k^I||^p + ||x_k^F||^p) \oplus (||y_k^T||^p + ||y||^p + ||y_k^F||^p)\}$$

It follows that $\sum_{k=1}^{\infty} || x_k + y_k||^p < \infty$.Thus, $(x_k + y_k) \in \ell_p^N(X)$.
Let $r \in \mathbb{R}$. We have

$$\sum_{k=1}^{\infty} ||rx_k||^p = |r|^p \sum_{k=1}^{\infty} ||x_k^T||^p + ||x_k^I||^p + ||x_k^F||^p) < \infty.$$

Thus, $rx_k \in \ell_p^N(X), \forall r \in \mathbb{R}$.
So, $\ell_p^N(X)$ is a subspace in $w^N(X)$ and thus neutrosophic normed linear space valued sequence space.

Theorem 4.2. Consider X that is a complete metric space. The space $\ell_p^N(X), 1 \le p < \infty$, is complete with the norm

$$||x|| = \{\sum_{k=1}^{\infty}(||x_k^T||^p + ||x_k^F||^p + ||x_k^I||^p)\}^{1/p},$$

where $x = (x_k) \in \ell_p^N(X)$, and $k \in \mathbb{N}$.

Proof. Consider $(x^{(n)})$ that is a Cauchy sequence in $\ell_p^N(X)$.

$$\text{Here } x^{(n)} = \left(x_k^{(n)}\right) = (x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \dots) \in l_p^N, \forall\, n \in \mathbb{N}.$$

Then for a given $\bar{\epsilon} > 0, \forall\, n_0 \in \mathbb{N}$ such that $||x^{(n)} - x^{(m)}|| = \{\sum((||x_k^{T^{(n)}} - x_k^{T^{(m)}}||^p + (||x_k^{I^{(n)}} - x_k^{I^{(m)}}||^p + (||x_k^{F^{(n)}} - x_k^{F^{(m)}}||^p)\}^{\frac{1}{p}} < \bar{\epsilon},\ \forall\, m, n \ge n_0$.

$$\Rightarrow ||x^{(n)} - x^{(m)}|| < \bar{\epsilon}$$

$\Rightarrow$ Sequence $\left(x_k^{(n)}\right)$ is Cauchy in X for every k in $\mathbb{N}$.
As X is complete, $\exists x_k \in X$ such that

$$\left|\left|x_k^{(n)} - x_k\right|\right| \to \bar{0}, as\ n \to \infty, for\ each\ k.$$

$\Rightarrow (|||x_k^{T^{(n)}} - x_k^T|||_2^\alpha) + (|||x_k^{F^{(n)}} - x_k^F|||_2^\alpha) + (|||x_k^{I^{(n)}} - x_k^I|||_2^\alpha) \to 0,$
$as\ n \to \infty$, for every $\alpha \in (0, 1]$.
$as\ n \to \infty$, for every $\alpha \in (0, 1]$.
Here sequence $x^{(n)}$ is also Cauchy. So, for every $\bar{\epsilon} > 0, \exists\ n_0 = n_0(\bar{\epsilon})$ such that

$$\{\sum((||x_k^{T^{(n)}} - x_k^{T^{(m)}}||^p + (||x_k^{I^{(n)}} - x_k^{I^{(m)}}||^p + (||x_k^{F^{(n)}} - x_k^{F^{(m)}}||^p)\}^{1/p} < \bar{\epsilon},$$

$$\Rightarrow [\sum\{((|||x_k^{T^{(n)}} - x_k^T|||_2^\alpha)^p + (|||x_k^{F^{(n)}} - x_k^F|||_2^\alpha)^p + (|||x_k^{I^{(n)}} - x_k^I|||_2^\alpha)^p]^{\frac{1}{p}} < \epsilon \text{ for each } \alpha \in (0, 1].$$

Now fix $n \ge n_0$ and let $m \to \infty$, to have
$[\sum\{((|||x_k^{T^{(n)}} - x_k^T|||_2^\alpha)^p + (|||x_k^{F^{(n)}} - x_k^F|||_2^\alpha)^p + (|||x_k^{I^{(n)}} - x_k^I|||_2^\alpha)^p]^{\frac{1}{p}} < \epsilon,$
for all $n \ge n_0$ and $\alpha \in (0,1]$.
=> $[\sum((||x_k^{T^{(n)}} - x_k^T||^p + (||x_k^{I^{(n)}} - x_k^I||^p + (||x_k^{F^{(n)}} - x_k^F||^p)\}^{1/p} < \bar{\epsilon}, \forall\, n \ge n_0$. ... ...(1)
=> $\{\sum(||x^{T^{(n)}} - x||)^p + (||x^{F^{(n)}} - x||)^p + (||x^{I^{(n)}} - x||)^p\}^{1/p} \le \bar{\epsilon},\ \forall\, n \ge n_0$, where $x = (x_k)$ ... ...(2)
Hence, $x^{(n)} \to x$, as $n \to \infty$.
Now we show that $x = (x_k) \in \ell_p^N(x)$.
From (2), we have, $\{\sum(||x^{T^{(n)}} - x||)^p + (||x^{F^{(n)}} - x||)^p + (||x^{I^{(n)}} - x||)^p\} \le \bar{\epsilon}^p,$

$$\Rightarrow \left(x^{(n)} - x\right) \in l_p^N(x).$$

Here $x = x^{(n)} + \left(x - x^{(n)}\right)$. Thus by Minkowski inequality and using (1), we get

$$\{\sum_{k=1}^{\infty}||x||^{p}\}^{\frac{1}{p}} \leq \{\sum_{k=1}^{\infty}||x^{(n)}|\quad |^{p}\}^{\frac{1}{p}} \oplus \{\sum_{k=1}^{\infty}||x-x^{n}||^{p}\}^{\frac{1}{p}}$$

$$\Rightarrow \{\sum_{k=1}^{\infty}(||x_k^T||^p + ||x_k^F||^p + ||x_k^I||^p)\}^{1/p} \leq (||x_k^T||^p + ||x_k^F||^p + ||x_k^I||^p))^{1/p} \oplus \bar{\epsilon}$$

$$\Rightarrow \sum_{k=1}^{\infty}[||x_k^{T^{(n)}}||^{p.1/p} + ||x_k^{F^{(n)}}||^{p.1/p} + ||x_k^{I^{(n)}}||^{p.1/p}] < \infty, since\ x^{(n)} = (x_k^{(n)}) \in l_p^N(X)$$

Thus $x \in \ell_p^N(X)$.
Thus, the result follows.

Theorem 4.3. Sequence space $\ell_p^N(X)$ is solid as well as monotone.

Proof. Let $x = (x_k)$ and $y = (y_k)$ be two sequences such that

$$\left|\left|x_k^T + x_k^F + x_k^I\right|\right| \leq \left|\left|y_k^T + y_k^F + y_k^I\right|\right| \ \forall\ k \in N \text{ and } (y_k) \in l_p^N(X).$$

$$\Rightarrow \sum(||x_k^T||^p + ||x_k^F||^p + ||x_k^I||^p) \leq \sum ||y_k^T||^p + ||y_k^F||^p + ||y_k^I||^p) < \infty,$$

$$\Rightarrow \sum(||x_k^T||^p + ||x_k^F||^p + ||x_k^I||^p) \leq \sum ||y_k^T||^p + ||y_k^F||^p + ||y_k^I||^p) < \infty,$$

Thus $x = (x_k) \in \ell_p^N(X)$ and $\ell_p^N(X)$ is solid.
Hence the space $\ell_p^N(X)$ is monotone.

Theorem 4.4. Space $\ell_p^N(X)$ cannot be convergent free.

Proof. Theorem is verified with the following example.

Example 4.5. Consider $x = (x_k)$ as a sequence given below:

$$x_k = \begin{cases} k^{-4}, for\ k\ even \\ 0, for\ k\ odd \end{cases} \dots \dots (3)$$

Consider *X* as a neutrosophic normed linear space and $z = (z_k) \in X$. Consider $\left|\left|z_k^T\right|\right|$ given below:

Take *k* that is a natural number and let $z_k \neq 0$, $||z_k^T||(t) = \begin{cases} \frac{5t}{|z_k|} - 4, for\ \frac{4|z_k|}{5} \leq t \leq |z_k| \\ 0, \text{otherwise} \end{cases} \dots \dots (4)$

and for $||z_k|| = 0, ||z_k||(t) = \begin{cases} 1, \text{for} t = 0 \\ 0, \text{otherwise} \end{cases} \dots \dots (5)$.

Using (3), we have for

k even and $x_k \neq 0$, $\left|\left|x_k^T\right|\right|(t) = \begin{cases} \frac{5t}{|x_k|} - 4, for\ \frac{4|x_k|}{5} \leq t \leq |x_k| = k^{-4}, \\ 0, \text{otherwise}, \end{cases} \dots \dots (6)$

and for *k* odd,

$$||x_k|| = 0, ||x_k||(t) = \begin{cases} 1, for\ t = 0, \\ 0, \text{otherwise.} \end{cases} \dots \dots (7)$$

Again, for each $\alpha \in (0,1]$, we have

$$[||x_k^T|| = \begin{cases} [\frac{\alpha + 4}{5} k^{-4}, k^{-4}], for\ k\ even \\ [0,0], for\ k\ odd \end{cases}$$

Hence for each $\alpha \in (0,1]$,

$\sum_{k=1}^{\infty}[|||x_k^T|||_2^{\alpha}]^p = [\sum_{k=1}^{\infty}(\frac{1}{k^4})^p - \sum_{k=0}^{\infty}[\frac{1}{(2k+1)^4}] < \infty$,

$\Rightarrow \sum_{k=1}^{\infty}||x_k^T||^p < \infty$.

Similarly, we can show that

$\sum_{k=1}^{\infty}||x_k^F||^p < \infty$ and $\sum_{k=1}^{\infty}||x_k^I||^p < \infty$.

Thus, $x = (x_k) \in \ell_p^N(X)$.

Let $y = (y_k)$ be this sequence:

$$y_k = \begin{cases} k^{-\frac{1}{p}}, for\ k\ even, \\ 0, for\ k\ odd. \end{cases}$$

Then for *k* even and using (4), we have

$$||y_k^T||(t) = \begin{cases} \frac{5t}{|y_k|} - 4, for\ \frac{4|y_k|}{5} \leq t \leq |y_k| = k^{-\frac{1}{p}}, \\ 0, \text{otherwise.} \end{cases}$$

Further when k is odd, $||y_k||(t) = \begin{cases} 1, \text{when } t = 0, \\ 0, \text{otherwise.} \end{cases}$
Further, every $\alpha \in (0, 1]$ gives

$$[||y_k^T||]^\alpha = \begin{cases} \left[\left(\frac{\alpha+4}{5}\right) k^{-\frac{1}{p}}, k^{-\frac{1}{p}}\right], \text{for k even,} \\ [0,0], \text{for k odd.} \end{cases}$$

Thus, for every $\alpha \in (0, 1]$,
$\sum_{k=1}^{\infty}[|||y_k^T|||_2^\alpha]^p = \sum_{k=1}^{\infty}(k^{-\frac{1}{p}})^p - \sum_{k=0}^{\infty}\{(2k+1)^{-\frac{1}{p}}\}^p$,
which is unbounded.
$\Rightarrow \sum_{k=1}^{\infty} ||y_k^T||^p$ *is unbounded.*
Thus $y = (y_k) \notin \ell_p^N(x)$ and $l_p^X(X)$ is not convergence free.

Theorem 4.6. Symmetricity of $\ell_p^N(X)$ holds.

Proof. Consider $x = (x_k) \in \ell_p^N(X)$.
Consider $y = (y_k)$ as rearrangement of $x = (x_k)$where $x_k = y_{m_k}$ *for each* $k \in \mathbb{N}$.
Then, $\sum ||y_{m_k}^T||^p = \sum ||x_k^T||^p < \infty$.
Similarly, we can show that $\sum ||x_k^F||^p < \infty$.
and $\sum ||x_k^I||^p < \infty$.
Thus, $y = (y_k) \in \ell_p^N(X)$.
Thus, the result follows.

Theorem 4.7. $\ell_p^N(X) \subseteq \ell_q^N(X)$, for $1 \leq p < q < \infty$.

Proof. Let $x = (x_k) \in \ell_p^N(X)$. Then we have
$\sum ||x_k^T||^p < \infty \Longrightarrow \sum[|||x_k^T|||_2^\alpha]^p < \infty$, for every $\alpha \in (0, 1]$.
Since, $\lim_{k\to\infty} ||x_k^T - 0|| = \bar{0}$ (as $(x_k) \in \ell_p^N(X)$), so $\exists\, n_0$ of $\mathbb{N}$ where
$||x_k^T - 0|| \leq \bar{1}$. Here every k is greater than equal to n_0.
Thus, $\sum ||x_k^T||^q = \sum_{k=1}^{n_0-1} ||x_k^T||^q \oplus \sum_{n=n_0}^{\infty} ||x_k^T||^q$.
Clearly,
$\sum_{k=n_0}^{\infty} ||x_k^T||^q \leq \sum_{k=1}^{\infty} ||x_k^T||^p < \infty$.
Further we have,
$\sum_{k=1}^{n-1} ||x_k^T||^q$ *is finite sum.*
Hence, $\sum ||x_k^T||^q < \infty$.
Similarly, we can have $\sum ||x_k^F||^q < \infty$. and $\sum ||x_k^I||^q < \infty$.
Thus, $x = (x_k) \in \ell_p^N(X)$and hence the result.

5. Conclusions

In this article, we introduced the notion of p - absolutely summable sequences ℓ_p^N, $1 \leq p < \infty$, of neutrosophic real numbers and investigated some of their algebraic and topological properties. We further examined some relationships involving this space. The methodology adopted to establish the results can be applied to study the class of p - absolutely summable double sequences. This space can be examined from a neutrosophic metric aspect. If we remove the indeterminacy, i.e., if we consider value of indeterminacy as zero, then we get intuitionistic fuzzy normed spaces, Again, if we remove the indeterminacy and falseness, i.e., if we consider both indeterminacy and falseness as zero, then we get fuzzy normed spaces

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