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Original Article

Γ-independent dominating graphs of paths and cycles

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Abstract

A set D of vertices in a graph G is an independent dominating set if D is a set of pairwise nonadjacent vertices of G such that every vertex of G not in D is adjacent to at least one vertex in D. An independent dominating set D is minimal if no proper subset of D is an independent dominating set. The upper independent domination number of a graph G, denoted by $\Gamma_i(G)$, is the maximum cardinality of a minimal independent dominating set of G. An independent dominating set D is an $\Gamma_i(G)$-set if D is of size $\Gamma_i(G)$. We introduce the Γ-independent dominating graph of G, denoted by $ID_\Gamma(G)$, as the graph whose vertex set is the set of all $\Gamma_i(G)$-sets, and two $\Gamma_i(G)$-sets are adjacent in $ID_\Gamma(G)$ if one can be obtained from the other by adding one vertex and removing another vertex of G. In this paper, we present the Γ-independent dominating graphs of all paths and cycles.

Keywords: path, cycle, upper independent dominating set, upper independent domination number, upper independent dominating graph

1. Introduction

Let $G = (V(G),E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A set $D \subseteq V(G)$ is a *dominating set* if every vertex not in D is adjacent to at least one vertex in D. A dominating set D is *minimal* if no proper subset of D is a dominating set. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a minimal dominating set of G or equivalently, the minimum cardinality of a dominating set in G. A $\gamma(G)$-set is a dominating set of G with cardinality $\gamma(G)$. For detailed literature on domination can be found in Haynes, Hedetniemi, & Slater (1998). For the notations and terminologies, which are not defined here, we in general follow West (2018).

Subramanian and Sridharan (2008) introduced the *gamma graph* of a graph G, denoted by $\gamma.G$, whose vertices correspond to $\gamma(G)$-sets. Two $\gamma(G)$-sets D_1 and D_2 are adjacent in $\gamma.G$ if $D_1 = (D_2 \setminus \{u\}) \cup \{v\}$ for some vertices $u \in D_2$ and $v \notin D_2$, i.e., they differ by exactly one vertex. In 2009, they proved that trees and unicyclic graphs are gamma graphs (Sridharan & Subramanian, 2009). Lakshmanan and Vijayakumar (2010) determined the clique numbers of a graph and its gamma graph. They also studied their independence numbers.

Fricke, Hedetniemi, Hedetniemi, and Hutson (2011) also defined the *gamma graph* of G with slightly different meaning, and denoted this gamma graph by $G(\gamma)$. The vertex set of $G(\gamma)$ is the same as one of $\gamma.G$. Two $\gamma(G)$-sets D_1 and D_2 are adjacent in $G(\gamma)$ if $D_1 = (D_2 \setminus \{u\}) \cup \{v\}$ for some vertices $u \in D_2$ and $v \notin D_2$, together with u must be adjacent to v in G. They provided the structural properties of $G(\gamma)$.

Haas and Seyffarth (2014) defined a *k-dominating graph* of G, denoted by $D_k(G)$, as the graph whose vertex set contains all dominating sets D of G that have cardinality at most k. Two dominating sets are adjacent in $D_k(G)$ if one can be obtained from the other by either adding or deleting a single vertex. These authors studied the conditions under which such graphs are connected.

Several authors have explored the idea of a gamma graph employing alternative forms of domination. For

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example, Wongsriya and Trakultraipruk (2017) defined the γ-*total dominating graph* $TD_\gamma(G)$ of G as the graph whose vertex set is the set of all minimal total dominating sets with the minimum cardinality. Two total dominating sets are adjacent in $TD_\gamma(G)$ if they differ by one vertex. They considered the γ-total dominating graphs of all paths and cycles. In the same year, Wongsriya, Kositwattanarerk, and Trakultraipruk (2017) defined the Γ-*total dominating graph* $TD_\Gamma(G)$ of G as the graph whose vertex set is the set of all minimal total dominating sets with the maximum cardinality. Two total dominating sets are adjacent in $TD_\Gamma(G)$ if they differ by one vertex. In their paper, they considered the Γ-total dominating graphs of some paths.

Eakavinrujee and Trakultraipruk (2018) defined the Γ-*paired dominating graph* $PD_\Gamma(G)$ using the same concept. They determined the Γ-paired dominating graphs of some paths. In 2022, they also defined the γ-*paired dominating graph* $PD_\gamma(G)$ of G. They presented two main results, namely the γ-paired dominating graphs of all paths and all cycles (Eakavinrujee & Trakultraipruk, 2022).

Sanguanpong and Trakultraipruk (2021) defined the Γ-*induced-paired dominating graph* $IPD_\Gamma(G)$ of G. In this paper, they determined the Γ-induced-paired dominating graphs of paths. Moreover, in 2022, they considered the γ-*induced-paired dominating graphs* $IPD_\gamma(G)$ of paths and cycles (Sanguanpong & Trakultraipruk, 2022).

A set of pairwise nonadjacent vertices is called an *independent set*. A set $D \subseteq V(G)$ is an *independent dominating set* of G if it is both an independent set and a dominating set of G. The theory of independent domination was formalized by Berge (1962) and Ore (1962). An independent dominating set D is *minimal* if no proper subset of D is an independent dominating set. The *independent domination number* of G, denoted by $\gamma_i(G)$, is the minimum cardinality of a minimal independent dominating set of G. An independent dominating set of cardinality $\gamma_i(G)$ is called an $\gamma_i(G)$-set. We (Samanmoo, Trakultraipruk, & Ananchuen, 2019) introduced the γ-*independent dominating graph* of G, denoted by $ID_\gamma(G)$, as the graph whose vertex set is the set of all $\gamma_i(G)$-sets. Two $\gamma_i(G)$-sets D_1 and D_2 are adjacent in $ID_\gamma(G)$ if $D_1=(D_2\setminus\{u\})\cup\{v\}$ for some vertices $u\in D_2$ and $v\notin D_2$. We considered the γ-independent dominating graphs of all paths and cycles. In the following year, Samanmoo and Trakultraipruk (2020) determined the γ-independent dominating graphs of gear graphs and lollipop graphs.

The *upper independent domination number* of G, denoted by $\Gamma_i(G)$, is the maximum cardinality of a minimal independent dominating set of G. A minimal independent dominating set of cardinality $\Gamma_i(G)$ is called an $\Gamma_i(G)$-set (upper independent dominating set of G). In this paper, we introduce the Γ-*independent dominating graph* of G, denoted by $ID_\Gamma(G)$, as the graph whose vertex set is the set of all $\Gamma_i(G)$-sets. Two $\Gamma_i(G)$-sets D_1 and D_2 are adjacent in $ID_\Gamma(G)$ if $D_1=(D_2\setminus\{u\})\cup\{v\}$ for some vertices $u\in D_2$ and $v\notin D_2$. For instance, the Γ-independent dominating graphs of the path $P_6 = v_1v_2\cdots v_6$ and the cycle $C_5 = v_0v_1\cdots v_4v_0$ are shown in Figures 1 and 2, respectively.

For any vertex v in an independent dominating set D, we have $D\setminus\{v\}$ contains no vertex adjacent to v. Hence, all independent dominating sets are minimal. We then obtain the following proposition.

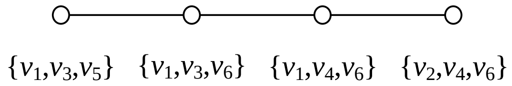

Figure 1. Graph $ID_\Gamma(P_6)$

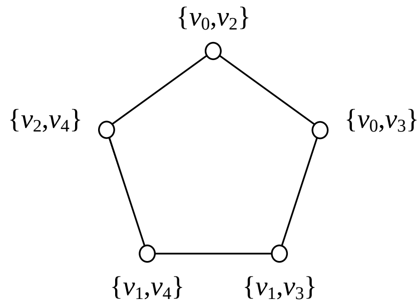

Figure 2. Graph $ID_\Gamma(C_5)$

Proposition 1. Let G be a graph. Then $\Gamma_i(G)$ equals the maximum cardinality of an independent dominating set.

Furthermore, we have that every vertex of degree one must be dominated by itself or its neighbor, and they cannot be both in any independent dominating set. Then we have the following proposition.

Proposition 2. Let G be a graph. If v is a vertex of degree one in G, then every $\Gamma_i(G)$-set must contain v or its neighbor.

2. Results

2.1 The Γ-independent dominating graphs of all paths

Let n be a positive integer and $P_n = v_1v_2\cdots v_n$ be a path, where v_i is a vertex for all i. In this section, we consider the Γ-independent dominating graph of P_n in two cases. If n is odd, then the Γ-independent dominating graph of this path is a trivial graph. If n is even, say $n = 2k$, for some positive integer k, then the Γ-independent dominating graph of P_n is a path with $k + 1$ vertices

To find the upper independent domination number of a path, by Proposition 1, it suffices to find the biggest size of an independent dominating set of that path. Moreover, any two adjacent vertices in this path cannot be both in the same independent dominating set, so we get the following proposition.

Proposition 3. Let $n \geq 1$ be an integer. Then $\Gamma_i(P_n) = \left\lceil \frac{n}{2} \right\rceil$.

Theorem 4. Let $k \geq 0$ be an integer. Then $ID_\Gamma(P_{2k+1}) \cong P_1$.

Proof. By Proposition 3, we have $\Gamma_i(P_{2k+1}) = k + 1$. It is obvious that there is only one $\Gamma_i(P_{2k+1})$-set, which is $\{v_1,v_3,\ldots,v_{2k-1},v_{2k+1}\}$. This completes the proof.

Lemma 5. Let $k \geq 1$ be an integer. Then there is only one $\Gamma_i(P_{2k})$-set containing v_{2k-1}, and this set has degree one in $ID_\Gamma(P_{2k})$.

Proof. If $k = 1$, there is only one $\Gamma_i(P_2)$-set containing v_1, which is $\{v_1\}$, and it is adjacent to only the set $\{v_2\}$ in $ID_\Gamma(P_2)$. Now suppose $k \geq 2$. Note that $\Gamma_i(P_{2k}) = k$. If v_{2k-1} is in an $\Gamma_i(P_{2k})$-set, the other $k-1$ vertices in this $\Gamma_i(P_{2k})$-set must come from $\{v_1,v_2,\ldots,v_{2k-3}\}$. We may consider these $2k-3$ vertices as a path. Since $\Gamma_i(P_{2k-3}) = k-1$, this $\Gamma_i(P_{2k})$-set is a union of an $\Gamma_i(P_{2k-3})$-set and $\{v_{2k-1}\}$. By Theorem 4, there is only one $\Gamma_i(P_{2k-3})$-set, which is $D = \{v_1,v_3,\ldots,v_{2k-3}\}$. We have $D' = D \cup \{v_{2k-1}\}$ is the unique $\Gamma_i(P_{2k})$-set that contains v_{2k-1}. By Proposition 2, each $\Gamma_i(P_{2k})$-set must contain either v_{2k-1} or v_{2k}. Hence all neighbors of D' have to contain v_{2k}. Then $D \cup \{v_{2k}\}$ is the only neighbor of D' in $ID_\Gamma(P_{2k})$, so it is of degree one.

Theorem 6. Let $k \geq 1$ be an integer. Then $ID_\Gamma(P_{2k}) \cong P_{k+1}$.

Proof. We prove by induction on k. For $k = 1$, there are two $\Gamma_i(P_2)$-sets, which are $\{v_1\}$ and $\{v_2\}$, so $ID_\Gamma(P_2) \cong P_2$.

Let $k \geq 1$. We assume that $ID_\Gamma(P_{2k}) \cong P_{k+1} \cong D_1D_2\cdots D_{k+1}$, where D_i is an $\Gamma_i(P_{2k})$-set for all i. By Lemma 5, without loss of generality, we may assume that D_{k+1} contains v_{2k-1}, and the other $\Gamma_i(P_{2k})$-sets contain v_{2k}. We will prove that $ID_\Gamma(P_{2k+2}) \cong P_{k+2}$. Recall that $\Gamma_i(P_{2k+2}) = k+1$, and each $\Gamma_i(P_{2k+2})$-set contains exactly one of v_{2k+1} and v_{2k+2}. We first consider all $\Gamma_i(P_{2k+2})$-sets that contain v_{2k+2}. Since v_{2k+2} is adjacent to only v_{2k+1}, the other k vertices in this $\Gamma_i(P_{2k+2})$-set must come from $\{v_1,v_2,\ldots,v_{2k}\}$. Since $\Gamma_i(P_{2k}) = k$, these k dominating vertices form an $\Gamma_i(P_{2k})$-set. Then such an $\Gamma_i(P_{2k+2})$-set is a union of an $\Gamma_i(P_{2k})$-set and $\{v_{2k+2}\}$. For $i \in \{1,2,\ldots,k+1\}$, let $D'_i = D_i \cup \{v_{2k+2}\}$. By the induction hypothesis, $D'_1,D'_2,\ldots,D'_{k+1}$ are all $\Gamma_i(P_{2k+2})$-sets containing v_{2k+2}, and they form a path $D'_1D'_2\cdots D'_{k+1}$ in $ID_\Gamma(P_{2k+2})$. We next consider all $\Gamma_i(P_{2k+2})$-sets that contain v_{2k+1}. Recall that $D'_{k+1} = D_{k+1} \cup \{v_{2k+2}\}$, where D_{k+1} is the unique $\Gamma_i(P_{2k})$-set containing v_{2k-1}. Let $D'_{k+2} = D_{k+1} \cup \{v_{2k+1}\}$. Then D'_{k+2} is an $\Gamma_i(P_{2k+2})$-set. By Lemma 5, it is the unique $\Gamma_i(P_{2k+2})$-set containing v_{2k+1}, and it is adjacent to only D'_{k+1}. This completes the proof.

2.2 The Γ-independent dominating graphs of all cycles

Let $n \geq 3$ be a positive integer and $C_n = v_0,e_0,v_1,e_1,\ldots,v_{n-1},e_{n-1},v_0$ be a cycle, where v_i is a vertex and e_i is an edge for all i. In this section, we consider the Γ-independent dominating graph of C_n in two cases. If n is an even number, then the Γ-independent dominating graph of this cycle contains only two nonadjacent vertices. If n is an odd number, then the Γ-independent dominating graph of C_n is a cycle with n vertices.

We have that any two adjacent vertices in a cycle cannot be both in an independent dominating set, so we get the following proposition.

Proposition 7. Let $n \geq 3$ be an integer. Then $\Gamma_i(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$.

Theorem 8. Let $k \geq 2$ be an integer. Then $ID_\Gamma(C_{2k}) \cong 2P_1$.

Proof. By Proposition 7, we have $\Gamma_i(C_{2k}) = k$. It is obvious that there are two $\Gamma_i(C_{2k})$-sets, which are $\{v_0,v_2,\ldots,v_{2k-2}\}$ and $\{v_1,v_3,\ldots,v_{2k-1}\}$. Since $k \geq 2$, these two sets contain at least two vertices, and they are disjoint. Thus, they are not adjacent in $ID_\Gamma(C_{2k})$.

For convenience, in the proof of the following theorem, we always skip the modulo $2k+1$ in the subscriptions of vertices, edges, and independent dominating sets. For example, we write $v_{i+1(\text{mod } 2k+1)}$ as v_{i+1}.

Theorem 9. Let $k \geq 1$ be an integer. Then $ID_\Gamma(C_{2k+1}) \cong C_{2k+1}$.

Proof. We claim that the number of $\Gamma_i(C_{2k+1})$-sets is $2k+1$, which equals the number of edges in C_{2k+1}. First, we show that for each $\Gamma_i(C_{2k+1})$-set, there is exactly one edge in C_{2k+1} whose endpoints are both not in that $\Gamma_i(C_{2k+1})$-set. Note that $\Gamma_i(C_{2k+1}) = k$. Since the k dominating vertices in each $\Gamma_i(C_{2k+1})$-set are incident to $2k$ difference edges, there is exactly one edge in C_{2k+1} not incident to any dominating vertices. We next show that for each edge in C_{2k+1}, there is only one $\Gamma_i(C_{2k+1})$-set which does not contain both of its endpoints. Let e be an edge of C_{2k+1}. If both endpoints of e are not in an $\Gamma_i(C_{2k+1})$-set, all k vertices in this $\Gamma_i(C_{2k+1})$-set come from the other $2k+1-2 = 2k-1$ vertices. We consider these $2k-1$ vertices as a path. By Theorem 4, there is only one independent dominating set of size k, which is an $\Gamma_i(P_{2k-1})$-set. Hence this $\Gamma_i(P_{2k-1})$-set is the unique $\Gamma_i(C_{2k+1})$-set, which does not contain both endpoints of e. Thus the total number of $\Gamma_i(C_{2k+1})$-sets is $2k+1$.

For each $i \in \{0,1,\ldots,2k\}$, let $D_i = \{v_{i+2m} : 1 \leq m \leq k\}$. It is easy to check that D_i is an $\Gamma_i(C_{2k+1})$-set, which does not contain the endpoints of e_i. For instance, the $\Gamma_i(C_{2k+1})$-sets D_0 and D_2 are shown in Figure. 3, respectively.

For a fix $i \in \{0,1,\ldots,2k\}$, we consider all neighbors of D_i. Note that D_i does not contain v_i and v_{i+1} but it contains v_{i-1} and v_{i+2}. To find a neighbor of D_i in $ID_\Gamma(C_{2k+1})$, we can only replace v_{i-1} by v_i or replace v_{i+2} by v_{i+1} from D_i. Then the set $D_i \backslash \{v_{i-1}\} \cup \{v_i\}$ is an $\Gamma_i(C_{2k+1})$-set, which does not contain v_{i-2} and v_{i-1}, so it is D_{i-2}. Similarly, we have the set $D_i \backslash \{v_{i+2}\} \cup \{v_{i+1}\}$ does not contain v_{i+2} and v_{i+3}, so it is D_{i+2}. Hence, D_i is adjacent to only D_{i+2} and D_{i-2}. Then $ID_\Gamma(C_{2k+1}) \cong D_0D_2 \cdots D_{2k}D_1D_3 \cdots D_{2k-1}D_0 \cong C_{2k+1}$.

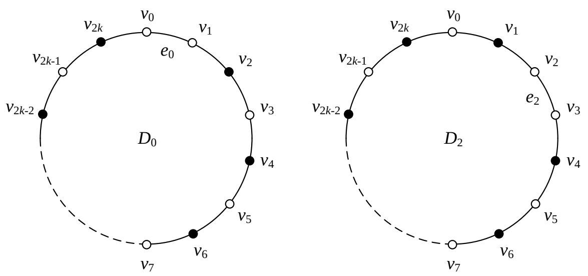

Figure 3. $\Gamma_i(C_{2k+1})$-sets D_0 and D_2 containing black vertices

3. Conclusions and Future Work

In this paper, we demonstrate that the Γ-independent dominating graph of P_{2k+1} is P_1, and for P_{2k} it is P_{k+1}. Additionally, we find that the Γ-independent dominating graph of C_{2k} is $2P_1$, and for C_{2k+1} it is the cycle itself. Further research will focus on determining the Γ-independent dominating graphs of other graphs.

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