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Original Article

Orthopairian fuzzy similarity based on below, above and centre fuzzy sets with applications to pattern recognition and identification of best ISP

Zahid Hussain[1*], Yasmeen Bano[1], Sahar Abbas[1],
Rashid Hussain[1], and Muhammad Alam[2]

[1] *Department of Mathematical Science, Karakoram International University, Gilgit-Baltistan, 15100 Pakistan*

[2] *Department of Earth Science, Karakoram International University, Gilgit-Baltistan, 15100 Pakistan*

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Abstract

Similarity measure plays an important role when estimating the degree of resemblance between two sets or objects. A variety of similarity measures are suggested in the literature in the context of fuzzy sets and their generalizations, but a similarity measure of q-rung orthopair fuzzy sets (q-ROFSs) based on below, above and centre fuzzy sets has not been considered so far. Therefore, in this paper, we propose a novel similarity measure based on the information carried by transforming q-rung orthopair fuzzy sets into their below, above and centre fuzzy sets to calculate the degree of similarity between two q-ROFSs. We also construct an axiomatic definition for the proposed similarity measure of q-ROFS. Furthermore, to show the competency, reliability and applicability of our proposed similarity measure, we present several examples related to pattern recognition and multicriteria decision making. Finally, we construct an algorithm for Orthopairian Portuguese interactive and multicriteria decision making (O-TODIM) based on our proposed similarity measure between q-ROFSs, to handle complex multicriteria decision making problems related to daily life. Our demonstration shows that the proposed method is reasonable and reliable in handling different problems related to daily life settings in the q-ROFSs environment.

Keywords: fuzzy set, q-rung orthopair fuzzy sets, similarity measures, pattern recognition, O-TODIM, multicriteria decision making

1. Introduction

Conventionally, two-way logic of yes-no type was used to model uncertain and incomplete information. For example, an assertion can either be true or false and nothing in between, and there is no place for even a little uncertainty. Fuzzy sets have been able to cope with these types of situations, give expert opinions and provide solution to many real world problems. Fuzzy set theory provides a strict mathematical framework in which vague conceptual phenomena can be precisely and rigorously studied. The word "fuzzy" means vague/ unclear/ imprecise/ ambiguous. In real world, there exists much fuzzy knowledge; knowledge that is vague, imprecise, uncertain, ambiguous, inexact or probabilistic in nature. Fuzzy set and its generalizations become an effective instrument to model incomplete information with elevated perfection. The first publications in fuzzy set theory by Zadeh (1965) and Goguen (1969) showed the intention of the authors to generalize the classical notion of a set and a proposition to accommodate fuzziness in the sense that it is contained in human language, that is, in human judgment, evaluation, and decisions.

In fuzzy set theory, the characteristic function is generalized to a membership function that assigns every x in X a value from the unit interval $[0,1]$ instead of being assigned from the two-element set $\{0,1\}$. The non-membership degree

*Corresponding author
Email address: zahid.hussain@kiu.edu.pk

is given by 1-membership. There are many situations in daily life where the non-membership degree is not considered the complement of membership but as some hesitancy degree. Therefore, Atanassov (1986) launched the concept of Intuitionistic fuzzy sets (IFSs), which includes a degree of belonging, a degree of non-belonging, and a degree of irresolution, to make this extension more applicable and useful. Intuitionistic fuzzy sets have numerous applications in many areas including pattern recognition (Hung & Yang, 2004), group decision making (Xu & Wang, 2016), IFS multi-objective mathematical programming (Mahapatra, 2006), an approach of IFS in medical diagnosis (De, 2001), and a clustering algorithm based on IFS (Xu, 2008). Similarity measure is an important tool to compare two objects. Similarity between two IFSs can be based on Sugeno integral with application to pattern recognition (Hung & Yang, 2004). A similarity measure is given by Kaufman and Rousseeuw (1991), and a new similarity/distance measure between intuitionistic fuzzy sets based on the transformed isosceles triangles and its applications to pattern recognition by Jiang and Jin (2019), relation between similarity measure is put forwarded by Liang and Shi (2003), similarity measure induced by Hausdorff distance is coined by Hung and Yang (2004), similarity between vague sets is given in He, Li, Qin, and Meng (2020); Hwang and Yang (2013), a possible and necessary inclusion of intuitionistic fuzzy sets is suggested (Grzegorzewski, 2011), a new construction for similarity measures between intuitionistic fuzzy sets based on lower, upper and middle fuzzy sets (Hwang & Yang, 2013), new similarity measures of intuitionistic fuzzy sets based on the Jaccard index with its application to clustering (Hwang, Yang & Hung, 2018) and similarity measures of intuitionistic fuzzy sets based on Hausdorff distance has been proposed (Hung & Yang, 2004).

Many extensions and generalizations of fuzzy sets have been made by researchers including Pythagorean fuzzy sets (PFSs) by (Yager, 2013) and (Yager & Abbasov, 2013), which are comparatively better than the IFSs. The characterization sof IFSs and PFSs are similar but they differ in the respective constraints $\mu(x) + \nu(x) \le 1$ and $\mu^2(x) + \nu^2(x) \le 1$. Therefore, IFSs now become the subsets of PFSs and PFSs can model uncertain situation better than the IFSs. Distance and similarity measures of Pythagorean fuzzy sets based on the Hausdorff metric with application to fuzzy TOPSIS (Hussain & Yang, 2019), Pythagorean Fuzzy LINMAP Method Based on the Entropy Theory for Railway Project Investment Decision Making (Xue, Xu, Zhang, & Tian, 2018) and Fuzzy entropy for Pythagorean fuzzy sets with application to multicriteria decision making (Yang & Hussain, 2018) have been reported. Most recently another amazing generalization of FS was coined by (Yager, 2017), q-rung orthopair fuzzy sets (q-ROFSs), which model uncertain and incomplete information better than either IFSs and PFSs with high accuracy. The constraint of q-ROFSs is $\mu^q(x) + \nu^q(x) \le 1$ and it covers a wider space than IFSs and PFSs. A q-rung Orthopair fuzzy multi-criteria group decision making method for supplier selection based on a novel distance measure is given (Adem & Boran, 2020) and information measures for q-ROFSs in the International Journal of Intelligent Systems is put forwarded (Peng & Liu, 2019) while similarity measures between q-ROFSs based on cosine functions are given by (Ping, Jie, Guiwu, & Cun, 2019). Similarity measures between q-ROFSs based on below, above and center fuzzy sets have not been considered so far. Therefore, in this manuscript, we propose a novel way to construct similarity measures between q-ROFSs based on the below, above and center fuzzy sets with applications to pattern recognition and multicriteria decision making with Orthopairian TODIM.

The rest of the paper is assembled as follows. In section 2, some basic concepts of intuitionistic fuzzy sets and Pythagorean fuzzy sets and q-ROFSs are briefly reviewed. We also concisely examine the q-ROFSs and an axiomatic definition of similarity measures. Section 3 is dedicated to construct new similarity measures of q-ROFSs based on the below, above and center fuzzy sets transmuted from the q-ROFSs. Section 4 is devoted to exhibit some examples and comparison among several proposed similarity measures of q-ROFSs. An application of the suggested methods in pattern recognition is also stated. In section 5, we construct Orthopairian fuzzy TODIM algorithm to apply our proposed similarity measure to solve complex daily life problems requiring multicriteria decision making. We wind up our investigation with a discussion in section 6.

2. Preliminaries

This section includes the review of basic notions of intuitionistic fuzzy sets, Pythagorean fuzzy sets and q-ROFSs.

Definition 1. (Atanassov, 1986). Let *X* be a universe of discourse. An IFS I in *X* is given by

$$\tilde{I} = \left\{ \left\langle x, \mu_{\tilde{I}}(x), \nu_{\tilde{I}}(x) \right\rangle : x \in X \right\}$$

where $\mu_{\tilde{I}} : X \to [0,1]$ denotes the degree of membership and $\nu_{\tilde{I}} : X \to [0,1]$ the degree of non-membership of the element $x \in X$ to the set $\tilde{I}$ with the condition that $0 \le \mu_{\tilde{I}}(x) + \nu_{\tilde{I}}(x) \le 1$. The degree of indeterminacy is $\pi_{\tilde{I}}(x) = 1 - \mu_{\tilde{I}}(x) - \nu_{\tilde{I}}(x)$.

Definition 2. (Yager, 2013). Let *X* be a universe of discourse. A Pythagorean fuzzy set (PFS) in *X* is given by

$$\tilde{P} = \left\{ \left\langle x, \mu_{\tilde{P}}(x), \nu_{\tilde{P}}(x) \right\rangle | \, x \in X \right\}$$

where $\mu_{\tilde{P}} : \mathrm{X} \to [0,1]$ denotes the degree of the membership and $\nu_{\tilde{P}} : X \to [0,1]$ denotes the degree of non-membership of the element $x \in X$ to the set $\tilde{P}$ respectively with the condition that $0 \le \mu_{\tilde{P}}^2(x) + \nu_{\tilde{P}}^2(x) \le 1$.

Definition 3. (Yager, 2017). Let *X* be a universe of discourse. A q-ROFS $\tilde{A}$ in *X* is given by

$$\tilde{A} = \left\{ < x,\ \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x) > | \ x \in X \right\}$$

where $\mu_{\tilde{A}}(x): X \to [0,1]$ represents the degree of membership and $\nu_{\tilde{A}}(x): X \to [0,1]$ indicates the degree of non-membership of the element $x \in X$ to the set $\tilde{A}$ with the condition that $0 \le \mu_{\tilde{A}}^q(x) + \nu_{\tilde{A}}^q(x) \le 1,\ q \ge 1.$ The degree of indeterminacy is $\pi_{\tilde{A}}(x) = \sqrt[q]{1 - \left(\mu_{\tilde{A}}^q(x) + \nu_{\tilde{A}}^q(x)\right)}$.

The main difference between IFSs, PFSs and q-ROFSs is in their corresponding constraints.

Definition 4. (Peng & Liu, 2019). Assume $\tilde{E}$ and $\tilde{F}$ be two q-ROFSs on universe of discourse *X*, then the following operations can be defined:

$(i)\ \tilde{E} \subseteq \tilde{F}\ iff\ \mu_{\tilde{E}}(x_i) \le \mu_{\tilde{F}}(x_i)$ and $\nu_{\tilde{E}}(x_i) \ge \nu_{\tilde{F}}(x_i),\ \ for\ \ i = 1, 2, ..., n.$

$(ii)\ \tilde{E} = \tilde{F}\ iff\ \mu_{\tilde{E}}(x_i) = \mu_{\tilde{F}}(x_i)$ and $\nu_{\tilde{E}}(x_i) = \nu_{\tilde{F}}(x_i),\ \ for\ \ i = 1, 2, ..., n.$

$(iii)\ \tilde{E} \cup \tilde{F} = \left\{ \left\langle x_i, \mu_{\tilde{E}}(x_i) \vee \mu_{\tilde{F}}(x_i), \nu_{\tilde{E}}(x_i) \wedge \nu_{\tilde{F}}(x_i) \right\rangle : x_i \in X \right\},\ i = 1, 2, ..., n.$

$(iv)\ \tilde{E} \cap \tilde{F} = \left\{ \left\langle x_i, \mu_{\tilde{E}}(x_i) \wedge \mu_{\tilde{F}}(x_i), \nu_{\tilde{E}}(x_i) \vee \nu_{\tilde{F}}(x_i) \right\rangle : x_i \in X \right\},\ i = 1, 2, ..., n.$

$(v)\ \tilde{E}^c = \left\{ \left\langle x_i, \mu_{\tilde{E}}(x_i), \nu_{\tilde{E}}(x_i) \right\rangle : x_i \in X \right\}.$

Now, we give the axiomatic definition of similarity between q-ROFSs.

Definition 5. Let $\tilde{E}$, $\tilde{F}$ and $\tilde{G}$ be any three q-ROFSs on a universal set *X*. A similarity measure $\tilde{S}(\tilde{E}, \tilde{F})$ is called an Orthopairian similarity for q-ROFSs if it satisfies the following axioms:

$(\tilde{S}1)\ \ 0 \le \tilde{S}(\tilde{E}, \tilde{F}) \le 1$;

$(\tilde{S}2)\ \ \tilde{S}(\tilde{E}, \tilde{F}) = 1,\ iff\ \ \tilde{E} = \tilde{F}$;

$(\tilde{S}3)\ \ \tilde{S}(\tilde{E}, \tilde{F}) = \tilde{S}(\tilde{F}, \tilde{E})$;

$(\tilde{S}4)\ \ If\ \ \tilde{E} \le \tilde{F} \le \tilde{G}$ then $\tilde{S}(\tilde{E}, \tilde{G}) \le \tilde{S}(\tilde{E}, \tilde{F})$ and $\tilde{S}(\tilde{E}, \tilde{G}) \le \tilde{S}(\tilde{F}, \tilde{G})$;

$(\tilde{S}5)\ \ \tilde{S}(\tilde{E}, \tilde{F}) = 0\ if\ \ \tilde{E} = X,\ \tilde{F} = \phi$ or $\tilde{F} = X\ ,\ \tilde{E} = \phi$.

Similarity between two IFSs $\tilde{E}$ and $\tilde{F}$ based on Lower, Upper and Middle fuzzy sets (LUMFSs) is given in (Hwang & Yang, 2013). We utilize a similar notion to define the similarity measures between two q-ROFSs based on Below, Above and Centre fuzzy sets (BACFSs) in the following section.

3. New Construction of Similarity Measures between q – ROFSs

Similarity measures are very useful tools to determine the resemblance between two objects. First, we use the notion given by (Hwang & Yang, 2013) to construct similarity measures between q-ROFSs. Then, we define new constructs of similarity measures between q-ROFSs based on BACFSs. The extended similarity measures between two q-rung Orthopair fuzzy sets $\tilde{E}$ and $\tilde{F}$ are defined as follows.

Suppose that $\tilde{E}$ and $\tilde{F}$ be any two q-ROFSs on a universal set *X* then,

$$\tilde{S}_{qRC}(\tilde{E}, \tilde{F}) = 1 - \frac{1}{2n} \sum_{i=1}^{n} \left| \left(\mu_{\tilde{E}}^q(x_i) - \nu_{\tilde{E}}^q(x_i) \right) - \left(\mu_{\tilde{F}}^q(x_i) - \nu_{\tilde{F}}^q(x_i) \right) \right| \tag{1}$$

$$\tilde{S}_{qRH}(\tilde{E}, \tilde{F}) = 1 - \frac{1}{2n} \sum_{i=1}^{n} \left(\left| \mu_{\tilde{E}}^q(x_i) - \mu_{\tilde{F}}^q(x_i) \right| + \left| \nu_{\tilde{E}}^q(x_i) - \nu_{\tilde{F}}^q(x_i) \right| \right) \tag{2}$$

$$\tilde{S}_{qRL}(\tilde{E}, \tilde{F}) = 1 - \frac{1}{4n} \sum_{i=1}^{n} \left| \left(\mu_{\tilde{E}}^q(x_i) - \nu_{\tilde{E}}^q(x_i) \right) - \left(\mu_{\tilde{F}}^q(x_i) - \nu_{\tilde{F}}^q(x_i) \right) \right| - \frac{1}{4n} \sum_{i=1}^{n} \left(\left| \mu_{\tilde{E}}^q(x_i) - \mu_{\tilde{F}}^q(x_i) \right| + \left| \nu_{\tilde{E}}^q(x_i) - \nu_{\tilde{F}}^q(x_i) \right| \right) \tag{3}$$

$$\tilde{S}_{qRO}(\tilde{E}, \tilde{F}) = 1 - \left[\frac{1}{2n} \sum_{i=1}^{n} \left(\left(\mu_{\tilde{E}}^q(x_i) - \mu_{\tilde{F}}^q(x_i) \right)^2 + \left(\nu_{\tilde{E}}^q(x_i) - \nu_{\tilde{F}}^q(x_i) \right)^2 \right) \right]^{\frac{1}{2}} \tag{4}$$

$$\tilde{S}_{qRDC}(\tilde{E}, \tilde{F}) = 1 - \frac{1}{\sqrt[p]{n}} \sqrt[p]{\sum_{i=1}^{n} \left| m_{\tilde{E}}(i) - m_{\tilde{F}}(i) \right|^p} \tag{5}$$

where $m_{\tilde{E}}(i) = \frac{1}{2}\left(\mu_{\tilde{E}}^q(x_i) + 1 - \nu_{\tilde{E}}^q(x_i) \right)$ *and* $m_{\tilde{F}}(i) = \frac{1}{2}\left(\mu_{\tilde{F}}^q(x_i) + 1 - \nu_{\tilde{F}}^q(x_i) \right),\ \ 1 \le p \le \infty.$

$$\tilde{S}_{qRHB}\left(\tilde{E},\tilde{F}\right)=\frac{1}{2}\left(\tilde{\rho}_{\mu}^{q}\left(\tilde{E},\tilde{F}\right)+\tilde{\rho}_{\nu}^{q}\left(\tilde{E},\tilde{F}\right)\right) \quad (6)$$

where $\tilde{\rho}_{\mu}^{q}\left(\tilde{E},\tilde{F}\right)=1-\frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}|\mu_{\tilde{E}}^{q}(x_i)-\mu_{\tilde{F}}^{q}(x_i)|^p}$ and $\tilde{\rho}_{\nu}^{q}\left(\tilde{E},\tilde{F}\right)=1-\frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}|\nu_{\tilde{E}}^{q}(x_i)-\nu_{\tilde{F}}^{q}(x_i)|^p}$

$$\tilde{S}_{qRe}^{p}\left(\tilde{E},\tilde{F}\right)=1-\frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}|\vartheta_{t\tilde{E}\tilde{F}}(i)+\vartheta_{f\tilde{E}\tilde{F}}(i)|^p} \quad (7)$$

where $\vartheta_{t\tilde{E}\tilde{F}}(i)=\frac{1}{2}\left|\mu_{\tilde{E}}^{q}(x_i)-\mu_{\tilde{F}}^{q}(x_i)\right|,\ \vartheta_{f\tilde{E}\tilde{F}}(i)=\frac{1}{2}\left|\left(1-\nu_{\tilde{E}}^{q}(x_i)\right)-\left(1-\nu_{\tilde{F}}^{q}(x_i)\right)\right|\ and\ 1\le p<\infty.$

$$\tilde{S}_{qRs}^{p}\left(\tilde{E},\tilde{F}\right)=1-\frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}\left(\phi_{s1}(i)+\phi_{s2}(i)\right)^p} \quad (8)$$

where

$\phi_{s1}(i)=\frac{1}{2}\left|m_{\tilde{E}1}(x_i)-m_{\tilde{F}1}(x_i)\right|and\ \phi_{s2}(i)=\frac{1}{2}\left|m_{\tilde{E}2}(x_i)-m_{\tilde{F}2}(x_i)\right|\ with\ m_{\tilde{E}1}(i)=\frac{1}{2}\left(\mu_{\tilde{E}}^{q}(x_i)+m_{\tilde{E}}(i)\right),$

$m_{\tilde{E}2}(i)=\frac{1}{2}\left(m_{\tilde{E}}^{q}(x_i)+1-\nu_{\tilde{E}}^{q}(i)\right),\ m_{\tilde{F}1}(i)=\frac{1}{2}\left(\mu_{\tilde{F}}^{q}(x_i)+m_{\tilde{F}}(i)\right)and\ m_{\tilde{F}2}(i)=\frac{1}{2}\left(m_{\tilde{F}}(x_i)+1-\nu_{\tilde{F}}^{q}(i)\right).$

$$\tilde{S}_{qRh}^{p}\left(\tilde{E},\tilde{F}\right)=1-\frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}\left(\sum_{m=1}^{3}\omega_m\tilde{\varphi}(i)\right)^p} \quad (9)$$

where $\tilde{\varphi}_1(i)=\tilde{\varphi}_{s1}(i)+\tilde{\varphi}_{s2}(i),\tilde{\varphi}_2(i)=|m_{\tilde{E}}(x_i)-m_{\tilde{F}}(x_i)|,\ \tilde{\varphi}_3(i)=max\left(l_{\tilde{E}}(i),l_{\tilde{F}}(i)\right)-min\left(l_{\tilde{E}}(i),l_{\tilde{F}}(i)\right)$ with $\omega_m=\frac{1}{3},\ l_{\tilde{E}}(i)=\frac{1}{2}\left(1-\nu_{\tilde{E}}^{q}(x_i)-\mu_{\tilde{E}}^{q}(x_i)\right)$ and $l_{\tilde{F}}(i)=\frac{1}{2}\left(1-\nu_{\tilde{F}}^{q}(x_i)-\mu_{\tilde{F}}^{q}(x_i)\right).$

Now, we construct a similarity between two q-ROFSs $\tilde{E}$ and $\tilde{F}$ based on below, above center fuzzy sets utilizing (1) - (9). Assume that $\tilde{E}=\left\{\left\langle x,\mu_{\tilde{E}}(x),\nu_{\tilde{E}}(x)\right\rangle : x\in X\right\}$ is a q–ROFS with $\mu_{\tilde{E}}^{q}(x)+\nu_{\tilde{E}}^{q}(x)\le 1,$ we first establish the below fuzzy set $\tilde{E}^b$ and the above fuzzy set $\tilde{E}^a$ to the q – ROFS according to (Grzegorzewski, 2011) and (Hwang & Yang, 2013) respectively as follows: $\tilde{E}^b=\left\{\left\langle x,\mu_{\tilde{E}^b}(x)\right\rangle : x\in X\right\}$, where $\mu_{\tilde{E}^b}(x)=\mu_{\tilde{E}}^{q}(x)$ and $\tilde{E}^a=\left\{\left\langle x,\mu_{\tilde{E}^a}(x)\right\rangle : x\in X\right\}$ with $\mu_{\tilde{E}^a}(x)=\mu_{\tilde{E}}^{q}(x)+\pi_{\tilde{E}}^{q}(x)=1-\nu_{\tilde{E}}^{q}(x).$ Furthermore, we interpret a center fuzzy set $\tilde{E}^c$ to the q–ROFS as follows: $\tilde{E}^c=\left\{\left\langle x,\mu_{\tilde{E}^c}(x)\right\rangle : x\in X\right\}$ where $\mu_{\tilde{E}^c}(x)=\frac{1}{2}\left(\mu_{\tilde{E}}^{q}(x)+1-\nu_{\tilde{E}}^{q}(x)\right)$

Now, we extend the above mentioned similarity measures (1)-(9) between two q-ROFSs to construct new similarity measures of q-ROFSs based on above, below and center fuzzy sets. Assume that the similarity measure $\tilde{S}\left(\tilde{E},\tilde{F}\right)$ between any two q-ROFSs $\tilde{E}$ and $\tilde{F}$ satisfies the conditions of Definition 4 in section 2. We can build a new similarity measure $\tilde{S}_{bac}\left(\tilde{E},\tilde{F}\right)$ based on the defined below, above and center fuzzy sets as follows:

$$\tilde{S}_{bac}\left(\tilde{E},\tilde{F}\right)=\frac{1}{3}\left(\tilde{S}\left(\tilde{E}^b,\tilde{F}^b\right)+\tilde{S}\left(\tilde{E}^a,\tilde{F}^a\right)+\tilde{S}\left(\tilde{E}^c,\tilde{F}^c\right)\right) \quad (10)$$

where $\tilde{E}^b=\left\{\left\langle x,\mu_{\tilde{E}^b}(x)\right\rangle : x\in X\right\}$ and $\mu_{\tilde{E}^b}(x)=\mu_{\tilde{E}}^{q}(x)$ and $\tilde{E}^a=\left\{\left\langle x,\mu_{\tilde{E}^a}(x)\right\rangle : x\in X\right\}$ with $\mu_{\tilde{E}^a}(x)=\mu_{\tilde{E}}^{q}(x)+\pi_{\tilde{E}}^{q}(x)=1-\nu_{\tilde{E}}^{q}(x).$ We next prove the suggested $\tilde{S}_{bac}\left(\tilde{E},\tilde{F}\right)$ is a similarity measure between two q – ROFSs $\tilde{E}$ and $\tilde{F}$.

Proposition 1. $0\le\tilde{S}_{bac}\left(\tilde{E},\tilde{F}\right)\le 1$.

Proof. Since $0\le\tilde{S}\left(\tilde{E}^b,\tilde{F}^b\right)\le 1,\ 0\le\tilde{S}\left(\tilde{E}^a,\tilde{F}^a\right)\le 1$ and $0\le\tilde{S}\left(\tilde{E}^c,\tilde{F}^c\right)\le 1$, apparently

$0\le\tilde{S}_{bac}\left(\tilde{E},\tilde{F}\right)\le 1$. Thus, the proposition is proved. □

Proposition 2. $\tilde{S}_{bac}\left(\tilde{E},\tilde{F}\right)=1\ iff\ \ \tilde{E}=\tilde{F}.$

Proof. Since $\tilde{S}\left(\tilde{E},\tilde{F}\right)$ is a similarity measure between E and F, by (S2) in Definition 4, we have that $\tilde{S}\left(\tilde{E},\tilde{F}\right)=1\, if\, \tilde{E}=\tilde{F}$. so, $\tilde{S}_{bac}\left(\tilde{E},\tilde{F}\right)=1\ if\ \tilde{E}^b=\tilde{F}^b, \tilde{E}^a=\tilde{F}^a$ and $\tilde{E}^c=\tilde{F}^c$ if $\mu^q_{\tilde{E}}\left(x_i\right)=\mu^q_{\tilde{F}}\left(x_i\right)$ $if\ \nu^q_{\tilde{E}}\left(x_i\right)=\nu^q_{\tilde{F}}\left(x_i\right)\ if\ \tilde{E}=\tilde{F}$. □

Proposition 3. $\tilde{S}_{bac}(\tilde{E},\tilde{F})=\tilde{S}_{bac}(\tilde{F},\tilde{E})$

Proof. Since $\tilde{S}(\tilde{E}^b,\tilde{F}^a)=\tilde{S}(\tilde{F}^b,\tilde{E}^b)$, $\tilde{S}(\tilde{E}^a,\tilde{F}^a)=\tilde{S}(\tilde{F}^a,\tilde{E}^a)$ and $\tilde{S}(\tilde{E}^c,\tilde{F}^c)=\tilde{S}(\tilde{F}^c,\tilde{E}^c)$, we have $\tilde{S}_{bac}\left(\tilde{E},\tilde{F}\right)=\frac{1}{3}\left(\tilde{S}\left(\tilde{E}^b,\tilde{F}^b\right)+\tilde{S}\left(\tilde{E}^a,\tilde{F}^a\right)+\tilde{S}\left(\tilde{E}^c,\tilde{F}^c\right)\right)=\frac{1}{3}\left(\tilde{S}\left(\tilde{F}^b,\tilde{E}^b\right)+\tilde{S}\left(\tilde{F}^a,\tilde{E}^a\right)+\tilde{S}\left(\tilde{F}^c,\tilde{E}^c\right)\right)=\tilde{S}_{bac}(\tilde{F},\tilde{E})$. □

Proposition 4. $\tilde{S}_{bac}\left(\tilde{E},\tilde{G}\right)\le\tilde{S}_{bac}\left(\tilde{E},\tilde{F}\right)$ and $\tilde{S}_{bac}\left(\tilde{E},\tilde{G}\right)\le\tilde{S}_{bac}\left(\tilde{F},\tilde{G}\right)$ if $\tilde{E}\subseteq\tilde{F}\subseteq\tilde{G}$.

Proof. If $\tilde{E}\subseteq\tilde{F}\subseteq\tilde{G}$ then for all *X*, we have $\mu^q_{\tilde{E}}\left(x\right)\le\mu^q_{\tilde{F}}\left(x\right)\le\mu^q_{\tilde{G}}\left(x\right)\Rightarrow 1-\mu^q_{\tilde{E}}\left(x\right)\ge 1-\mu^q_{\tilde{F}}\left(x\right)\ge 1-\mu^q_{\tilde{G}}\left(x\right)$, $1-\nu^q_{\tilde{E}}\left(x\right)\le 1-\nu^q_{\tilde{F}}\left(x\right)\le 1-\nu^q_{\tilde{G}}\left(x\right)$ and $1+\mu^q_{\tilde{E}}\left(x\right)-\nu^q_{\tilde{E}}\left(x\right)\le 1+\mu^q_{\tilde{F}}\left(x\right)-\nu^q_{\tilde{F}}\left(x\right)\le 1+\mu^q_{\tilde{G}}\left(x\right)-\nu^q_{\tilde{G}}\left(x\right)$ also $\frac{1}{2}\left(1+\mu^q_{\tilde{E}}\left(x\right)-\nu^q_{\tilde{E}}\left(x\right)\right)\le\frac{1}{2}\left(1+\mu^q_{\tilde{F}}\left(x\right)-\nu^q_{\tilde{F}}\left(x\right)\right)\le\frac{1}{2}\left(1+\mu^q_{\tilde{G}}\left(x\right)-\nu^q_{\tilde{G}}\left(x\right)\right)$,
which follows $\tilde{E}^b\subseteq\tilde{F}^b\subseteq\tilde{G}^b$, $\tilde{E}^a\subseteq\tilde{F}^a\subseteq\tilde{G}^a$ and $\tilde{E}^c\subseteq\tilde{F}^c\subseteq\tilde{G}^c$. Thus,
$\tilde{S}(\tilde{E}^b,\tilde{G}^b)\le\tilde{S}(\tilde{E}^b,\tilde{F}^b)$ $\tilde{S}(\tilde{E}^a,\tilde{G}^a)\le\tilde{S}(\tilde{E}^a,\tilde{F}^a)$ and $\tilde{S}(\tilde{E}^c,\tilde{G}^c)\le\tilde{S}(\tilde{E}^c,\tilde{F}^c)$. Hence $\tilde{S}_{bac}(\tilde{E},\tilde{G})\le\tilde{S}_{bac}(\tilde{E},\tilde{F})$.
Similarly, we have $\tilde{S}_{bac}(\tilde{E},\tilde{G})\le\tilde{S}_{bac}(\tilde{F},\tilde{G})$. □

Proposition 5. $\tilde{S}_{bac}(\tilde{E},\tilde{F})=0\, if\ \tilde{E}=X$ and $\tilde{F}=\varphi\, or\, \tilde{E}=\varphi$ and $\tilde{F}=X$, where *X* and *φ* are crisp sets.

Proof. If $\tilde{E}=X$ and $\tilde{F}=\varphi$ then $\tilde{E}^a=\tilde{E}^b=\tilde{E}^c=X$ and $\tilde{F}^a=\tilde{F}^b=\tilde{F}^c=\varphi$ by $\left(\tilde{S}5\right)$ of Definition 4,
since *X* and *φ* are crisps, we have $\tilde{S}(\tilde{E}^a,\tilde{F}^a)=0$ and $\tilde{S}(\tilde{E}^c,\tilde{F}^c)=0$. Thus
$\tilde{S}_{bac}\left(\tilde{E},\tilde{F}\right)=\frac{1}{3}\left(\tilde{S}\left(\tilde{E}^b,\tilde{F}^b\right)+\tilde{S}\left(\tilde{E}^a,\tilde{F}^a\right)+\tilde{S}\left(\tilde{E}^c,\tilde{F}^c\right)\right)=0$. Similarly, if $\tilde{E}=\varphi$ and $\tilde{F}=X$ then
$\tilde{S}_{bac}\left(\tilde{E},\tilde{F}\right)=\frac{1}{3}\left(\tilde{S}\left(\tilde{E}^b,\tilde{F}^b\right)+\tilde{S}\left(\tilde{E}^a,\tilde{F}^a\right)+\tilde{S}\left(\tilde{E}^c,\tilde{F}^c\right)\right)=0$. On the other hand, if $\tilde{S}_{bac}\left(\tilde{E},\tilde{F}\right)=0$ then
$\tilde{S}(\tilde{E}^b,\tilde{F}^b)=0$, $\tilde{S}(\tilde{E}^a,\tilde{F}^a)=0$ and $S(\tilde{E}^c,\tilde{F}^c)=0$. This implies that $\tilde{E}^a=\tilde{F}^b=\tilde{E}^c=X$ and $\tilde{F}^a=\tilde{E}^b=\tilde{F}^c=\varphi$ or $\tilde{E}^a=\tilde{F}^b=\tilde{E}^c=\varphi$ and $\tilde{F}^a=\tilde{E}^b=\tilde{F}^c=X$. Thus, we get $\tilde{E}=X$ and $\tilde{F}=\varphi$ or $\tilde{E}=\varphi$ and $\tilde{F}=X$. □

Evidently, we acquire the following theorem from propositions 1 to 5.

Theorem. $\tilde{S}_{bac}\left(\tilde{E},\ \tilde{F}\right)$ is a similarity measure if $\tilde{S}\ \left(\tilde{E},\ \tilde{F}\right)$ is a similarity measure between q- ROFS $\tilde{E}$ and $\tilde{F}$.

Next, we extend the similarity measures (1) – (9) between two q-ROFSs $\tilde{E}$ and $\tilde{F}$ to similarity measures between two q-ROFSs $\tilde{E}$ and $\tilde{F}$ based on below, above and center fuzzy sets as follows:

$$\tilde{S}_{bacC}(\tilde{E},\tilde{F})=1-\frac{1}{3n}\sum_{i=1}^{n}\left|\mu^q_{\tilde{E}}\left(x_i\right)-\mu^q_{\tilde{F}}\left(x_i\right)\right|+\left|\nu^q_{\tilde{E}}\left(x_i\right)-\nu^q_{\tilde{F}}\left(x_i\right)\right|+\frac{1}{2}\left|\mu^q_{\tilde{E}}\left(x_i\right)-\mu^q_{\tilde{F}}\left(x_i\right)+\nu^q_{\tilde{F}}\left(x_i\right)-\nu^q_{\tilde{E}}\left(x_i\right)\right| \tag{11}$$

$$\tilde{S}_{bacH}(\tilde{E},\tilde{F})=1-\frac{1}{3n}\sum_{i=1}^{n}\left|\mu^q_{\tilde{E}}\left(x_i\right)-\mu^q_{\tilde{F}}\left(x_i\right)\right|+\left|\nu^q_{\tilde{E}}\left(x_i\right)-\nu^q_{\tilde{F}}\left(x_i\right)\right|+\frac{1}{2}\left|\mu^q_{\tilde{E}}\left(x_i\right)-\mu^q_{\tilde{F}}\left(x_i\right)+\nu^q_{\tilde{F}}\left(x_i\right)-\nu^q_{\tilde{E}}\left(x_i\right)\right| \tag{12}$$

$$\tilde{S}_{bacL}(\tilde{E},\tilde{F})=1-\frac{1}{3n}\sum_{i=1}^{n}\left|\mu_{\tilde{E}}^{q}(x_i)-\mu_{\tilde{F}}^{q}(x_i)\right|+\left|\nu_{\tilde{E}}^{q}(x_i)-\nu_{\tilde{F}}^{q}(x_i)\right|+\frac{1}{2}\left|\mu_{\tilde{E}}^{q}(x_i)-\mu_{\tilde{F}}^{q}(x_i)+\nu_{\tilde{F}}^{q}(x_i)-\nu_{\tilde{E}}^{q}(x_i)\right| \quad (13)$$

$$\tilde{S}_{bacO}(\tilde{E},\tilde{F})=1-\frac{1}{3}\left(\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left|\mu_{\tilde{E}}^{q}(x_i)-\mu_{\tilde{F}}^{q}(x_i)\right|^2}+\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left|\nu_{\tilde{E}}^{q}(x_i)-\nu_{\tilde{F}}^{q}(x_i)\right|^2}+\sqrt{\frac{1}{4n}\sum_{i=1}^{n}\left|\mu_{\tilde{E}}^{q}(x_i)-\mu_{\tilde{F}}^{q}(x_i)+\nu_{\tilde{F}}^{q}(x_i)-\nu_{\tilde{E}}^{q}(x_i)\right|^2}\right) \quad (14)$$

$$\tilde{S}_{bacDC}(\tilde{E},\tilde{F})=1-\frac{1}{3}\left(\sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}\left|\mu_{\tilde{E}}^{q}(x_i)-\mu_{\tilde{F}}^{q}(x_i)\right|^p}+\sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}\left|\nu_{\tilde{E}}^{q}(x_i)-\nu_{\tilde{F}}^{q}(x_i)\right|^p}+\sqrt[p]{\frac{1}{2^p n}\sum_{i=1}^{n}\left|\mu_{\tilde{E}}^{q}(x_i)-\mu_{\tilde{F}}^{q}(x_i)+\nu_{\tilde{F}}^{q}(x_i)-\nu_{\tilde{E}}^{q}(x_i)\right|^p}\right) \quad (15)$$

$$\tilde{S}_{bacHB}(\tilde{E},\tilde{F})=1-\frac{1}{3}\left(\sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}\left|\mu_{\tilde{E}}^{q}(x_i)-\mu_{\tilde{F}}^{q}(x_i)\right|^p}+\sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}\left|\nu_{\tilde{E}}^{q}(x_i)-\nu_{\tilde{F}}^{q}(x_i)\right|^p}+\sqrt[p]{\frac{1}{2^p n}\sum_{i=1}^{n}\left|\mu_{\tilde{E}}^{q}(x_i)-\mu_{\tilde{F}}^{q}(x_i)+\nu_{\tilde{F}}^{q}(x_i)-\nu_{\tilde{E}}^{q}(x_i)\right|^p}\right) \quad (16)$$

$$\tilde{S}_{bace}^{p}(\tilde{E},\tilde{F})=1-\frac{1}{3}\left(\sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}\left|\mu_{\tilde{E}}^{q}(x_i)-\mu_{\tilde{F}}^{q}(x_i)\right|^p}+\sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}\left|\nu_{\tilde{E}}^{q}(x_i)-\nu_{\tilde{F}}^{q}(x_i)\right|^p}+\sqrt[p]{\frac{1}{2^p n}\sum_{i=1}^{n}\left|\mu_{\tilde{E}}^{q}(x_i)-\mu_{\tilde{F}}^{q}(x_i)+\nu_{\tilde{F}}^{q}(x_i)-\nu_{\tilde{E}}^{q}(x_i)\right|^p}\right) \quad (17)$$

$$\tilde{S}_{bacs}^{p}(\tilde{E},\tilde{F})=1-\frac{1}{3}\left(\sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}\left|\mu_{\tilde{E}}^{q}(x_i)-\mu_{\tilde{F}}^{q}(x_i)\right|^p}+\sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}\left|\nu_{\tilde{E}}^{q}(x_i)-\nu_{\tilde{F}}^{q}(x_i)\right|^p}+\sqrt[p]{\frac{1}{2^p n}\sum_{i=1}^{n}\left|\mu_{\tilde{E}}^{q}(x_i)-\mu_{\tilde{F}}^{q}(x_i)+\nu_{\tilde{F}}^{q}(x_i)-\nu_{\tilde{E}}^{q}(x_i)\right|^p}\right) \quad (18)$$

$$\tilde{S}_{bach}^{p}(\tilde{E},\tilde{F})=1-\frac{1}{3}\left(\sqrt[p]{\frac{2^p}{3^p n}\sum_{i=1}^{n}\left|\mu_{\tilde{E}}^{q}(x_i)-\mu_{\tilde{F}}^{q}(x_i)\right|^p}+\sqrt[p]{\frac{2^p}{3^p n}\sum_{i=1}^{n}\left|\nu_{\tilde{E}}^{q}(x_i)-\nu_{\tilde{F}}^{q}(x_i)\right|^p}+\sqrt[p]{\frac{1}{3^p n}\sum_{i=1}^{n}\left|\mu_{\tilde{E}}^{q}(x_i)-\mu_{\tilde{F}}^{q}(x_i)+\nu_{\tilde{F}}^{q}(x_i)-\nu_{\tilde{E}}^{q}(x_i)\right|^p}\right) \quad (19)$$

4. Numerical Results and Applications

In this section, we first give some numerical examples to show the validity of our proposed similarity measures (1) – (9) between q-ROFSs. Then, we present a few numerical examples to demonstrate our new construction of similarity measures (11) - (19) between q-ROFSs based on above, below and center fuzzy sets.

Example 1. Let us take six q-ROFSs. The numerical analysis results of similarity measures of q-ROFSs are shown in the Table 1 that demonstrates the similarity measure (1) – (9) of six different q-ROFSs. Clearly, the numerical simulations results in Table 1 show that there is no conflict in measuring similarity utilizing (2) – (7) between q-ROFSs except for (1), (8) and (9) having a little differences in a few places.

Table 1. Results of similarity measures (1) – (9) of q-ROFSs

	1	2	3	4	5	6
$\tilde{E}$	(x, 0.8, 0.7)	(x, 0.9, 0.5)	(x, 0.4, 0.9)	(x, 0.5, 0.4)	(x, 0.6, 0.6)	(x, 0.8, 0.5)
$\tilde{F}$	(x, 0.6, 0.9)	(x, 0.7, 0.8)	(x, 0.6, 0.8)	(x, 0.7, 0.3)	(x, 0.8, 0.5)	(x, 0.9, 0.4)
$\tilde{S}_{qRC}$	0.6590	0.6135	0.8155	0.8725	0.8065	0.8610
$\tilde{S}_{qRH}$	0.6590	0.6135	0.8155	0.8725	0.8065	0.8610
$\tilde{S}_{qRL}$	0.6590	0.6135	0.8155	0.8725	0.8065	0.8610
$\tilde{S}_{qRO}$	0.6590	0.6135	0.8127	0.8436	0.7810	0.8406
$\tilde{S}_{qRDC}$	0.6590	0.6135	0.8155	0.8725	0.8065	0.8610
$\tilde{S}_{qRHB}$	0.6590	0.6135	0.8155	0.8725	0.8065	0.8610
$\tilde{S}^{p}_{qRe}$	0.6590	0.6135	0.8155	0.8725	0.8065	0.8510
$\tilde{S}^{p}_{qRs}$	0.6590	0.6135	0.8155	0.8725	0.8065	0.8510
$\tilde{S}^{p}_{qRh}$	0.7576	0.7275	0.8533	0.8848	0.8368	0.8813

Next, we utilize our newly proposed methods of calculating similarity between q-ROFSs to check the reliability and reasonability applying on exhibit the numerical results of newly constructed similarity measures (11) - (19) of q-ROPFSs based on below, above and center fuzzy sets. The numerically calculated results show the reliability and suitability of our proposed method in Table 2 that shows the similarity measures (11) – (19) of q-ROFSs based on below, above and center fuzzy sets. There is no conflict in measuring the degree of similarity using (12), (13), (15) – (18) except for a little difference in similarity measures (11), (14) and (19) in a few places. The numerical analysis results show the validity and suitability of our proposed similarity measures (11) – (19).

Table 2. Similarity measures (11)–(19) of q-ROFS based on below, above and center fuzzy sets

	1	2	3	4	5	6
$\tilde{E}$	(x, 0.8, 0.7)	(x, 0.9, 0.5)	(x, 0.4, 0.9)	(x, 0.5, 0.4)	(x, 0.6, 0.6)	(x, 0.8, 0.5)
$\tilde{F}$	(x, 0.6, 0.9)	(x, 0.7, 0.8)	(x, 0.6, 0.8)	(x, 0.7, 0.3)	(x, 0.8, 0.5)	(x, 0.9, 0.4)
$\tilde{S}_{bacC}$	0.6590	0.6135	0.8155	0.8725	0.8065	0.8610
$\tilde{S}_{bacH}$	0.6590	0.6135	0.8155	0.8725	0.8065	0.8610
$\tilde{S}_{bacL}$	0.6590	0.6135	0.8155	0.8725	0.8065	0.8610
$\tilde{S}_{bacO}$	0.6590	0.6135	0.8126	0.8436	0.7810	0.8406
$\tilde{S}_{bacDC}$	0.6590	0.6135	0.8155	0.8725	0.8065	0.8610
$\tilde{S}_{bacHB}$	0.6590	0.6135	0.8155	0.8725	0.8065	0.8610
$\tilde{S}^{p}_{bace}$	0.6590	0.6135	0.8155	0.8725	0.8065	0.8610
$\tilde{S}^{p}_{bacs}$	0.6590	0.6135	0.8155	0.8725	0.8065	0.8610
$\tilde{S}^{p}_{bach}$	0.7727	0.7423	0.9734	0.9314	0.9114	0.9344

4.1 Application to pattern recognition

In this subsection, we utilize our proposed similarity measures (1) – (9) and (11) - (19) between q-ROFSs in pattern recognition to check the suitability and practical applicability of our proposed method.

Example 2. Let L_1 and L_2 be two patterns in the finite universe of discourse $X = \{x\}$

$$L_1 = \{(x, 0.75, 0.75)\} \; and \; L_2 = \{(x, 0.85, 0.65)\}$$

The sample K is represented by the q-ROFS as follows $K = \{(x, 0.95, 0.50)\}$. Our main objective is to classify the pattern K in one of the classes L_1 and L_2. According to the principle of maximum degree of similarity between q-ROFSs, the process of allocating the sample K to L_m is defined by the following relation

$$S_{q-ROFS,m} = \underset{1 \le j \le 2}{argmax}\left(S_{q-ROFS}\left(L_j, K\right)\right)$$

Utilizing the proposed similarity measures (1) – (9) and (11) - (19) between q-ROFSs to show the practical applicability in pattern recognition as follows:

$$\tilde{S}_C(L_1,K)=0.6338,\quad \tilde{S}_C(L_2,K)=0.8035,\quad \tilde{S}_{bacC}(L_1,K)=0.6338,\quad \tilde{S}_{bacC}(L_2,K)=0.8035$$
$$\tilde{S}_{qRH}(L_1,K)=0.6338,\quad \tilde{S}_{qRH}(L_2,K)=0.8035,\quad \tilde{S}_{bacH}(L_1,K)=0.6338,\quad \tilde{S}_{bacH}(L_2,K)=0.8035,$$
$$\tilde{S}_{qRL}(L_1,K)=0.6338,\quad \tilde{S}_{qRL}(L_2,K)=0.8035,\quad \tilde{S}_{bacL}(L_1,K)=0.6338,\quad \tilde{S}_{bacL}(L_2,K)=0.8035,$$
$$\tilde{S}_{qRO}(L_1,K)=0.6273,\quad \tilde{S}_{qRO}(L_2,K)=0.7980,\quad \tilde{S}_{bacO}(L_1,K)=0.6338,\quad \tilde{S}_{bacO}(L_2,K)=0.8035,$$
$$\tilde{S}_{qRDC}(L_1,K)=0.6338,\quad \tilde{S}_{qRDC}(L_2,K)=0.8035,\quad \tilde{S}_{bacDC}(L_1,K)=0.6338,\quad \tilde{S}_{bacDC}(L_2,K)=0.8035,$$
$$\tilde{S}_{qRHB}(L_1,K)=0.6338,\quad \tilde{S}_{qRHB}(L_2,K)=0.8035,\quad \tilde{S}_{bacHB}(L_1,K)=0.6338,\quad \tilde{S}_{bacHB}(L_2,K)=0.8035,$$
$$\tilde{S}^p_{qRe}(L_1,K)=0.6338,\quad \tilde{S}^p_{qRe}(L_2,K)=0.8035,\quad \tilde{S}^p_{bace}(L_1,K)=0.6338,\quad \tilde{S}^p_{bace}(L_2,K)=0.8035,$$
$$\tilde{S}^p_{qRs}(L_1,K)=0.4680,\quad \tilde{S}^p_{qRs}(L_2,K)=0.7170,\quad \tilde{S}^p_{bacs}(L_1,K)=0.6338,\quad \tilde{S}^p_{bacs}(L_2,K)=0.8035,$$
$$\tilde{S}^p_{qRh}(L_1,K)=0.6775,\quad \tilde{S}^p_{qRh}(L_2,K)=0.8245,\quad \tilde{S}^p_{bach}(L_1,K)=0.7558,\quad \tilde{S}^p_{bach}(L_2,K)=0.8690.$$

The above numerically computed results reflect intuitively that the pattern K belong to the sample L_2 according to the principle of maximum degree of similarity between q-ROFSs. All suggested similarity measures (1) – (9) and (11) - (19) between q-ROFSs unanimously agreed that the pattern K belongs to the pattern L_2.

5. q-Rung Orthopair Fuzzy TODIM Approach to Multi-Criteria Decision Making

Step 1: Let $\tilde{A}=\{\tilde{A}_1,\tilde{A}_2,...,\tilde{A}_i\}$ represent the set of alternatives and the set of criteria is represented by $\tilde{C}=\{\tilde{C}_1,\tilde{C}_2,...,\tilde{C}_j\}$. Identify the q-Rung Orthopair fuzzy decision matrix $\tilde{R}=(\mathrm{r}_{ij})_{m\times n}$ given by the DM in the MCDM problems, where r_{ij} is a q-ROFN. The decision matrix is constructed as follow:

$$\tilde{R}=(\mathrm{r}_{ij})_{m\times n}=\begin{array}{c|ccccc} & \tilde{C}_1 & \tilde{C}_2 & . & . & . & \tilde{C}_j \\ \hline \tilde{A}_1 & \mathrm{r}_{11} & \mathrm{r}_{12} & . & . & . & \mathrm{r}_{1j} \\ \tilde{A}_2 & \mathrm{r}_{21} & \mathrm{r}_{22} & . & . & . & \mathrm{r}_{2j} \\ . & . & . & . & & & . \\ . & . & . & & . & & . \\ . & . & . & & & . & . \\ \tilde{A}_i & \mathrm{r}_{i1} & \mathrm{r}_{i2} & . & . & . & \mathrm{r}_{ij} \end{array}$$

Step 2: Transform the decision matrix $\tilde{R}=(\mathrm{r}_{ij})_{m\times n}$ into a normalized q-ROFF decision matrix

$$\tilde{L}=(l_{ij})_{m\times n}=\begin{cases} r_{ij} & for \text{ beneficial attribute} \\ (r_{ij})^c & for \text{ cost attribute} \end{cases}$$

In this step, we transform the decision matrix $\tilde{R}=(\mathrm{r}_{ij})_{m\times n}$ into a normalized decision matrix. If the criteria are benefits then we write the original matrix, but if the criterion is a cost then we take its complement $(\mathrm{r}_{ij})^c$.

Step 3: Calculate the relative weight of each criterion $\tilde{C}_j$ using $\tilde{w}_{jr}=\tilde{w}_j/\tilde{w}_r$ where $\tilde{w}_j$ is weight of criterion $\tilde{C}_j$. In TODIM method, we choose the highest weight $\tilde{w}_r$ as a reference weight and divide the reference weight to all weights $\tilde{w}_j$.

$$\tilde{w}_r=max\left[\tilde{w}_j : j=1,2,3,...,n\right] and\ 0\le \tilde{w}_{jr}\le 1$$

Step 4: Calculate the dominance degree of each alternative $\tilde{A}_i$ over each alternative $\tilde{A}_t$ with respect to the criterion by $\tilde{C}_j$ using

$$\tilde{\phi}_j(\tilde{A}_i,\tilde{A}_t)=\begin{cases} \sqrt{\dfrac{\tilde{w}_{rj}\, d(\tilde{I}_{ij},\tilde{I}_{tj})}{\sum_{j=1}^{n}\tilde{w}_{jr}}} & if\ \ \tilde{I}_{ij}>\tilde{I}_{tj} \\ 0 & if\ \ \tilde{I}_{ij}=\tilde{I}_{tj} \\ -\dfrac{1}{\tilde{\theta}}\sqrt{\dfrac{\sum_{j=1}^{n}\tilde{w}_{rj}\, d(\tilde{I}_{ij},\tilde{I}_{tj})}{\tilde{w}_{jr}}} & if\ \ \tilde{I}_{ij}<\tilde{I}_{tj} \end{cases}$$

Here, $\tilde{\phi}_j\left(\tilde{A}_i,\tilde{A}_t\right)$ signifies dominance degree of the alternative $\tilde{A}_i$ over individually alternative $\tilde{A}$ with respect to the criterion by $\tilde{C}_j$ and equating alternatives i with alternatives t. $\tilde{\theta}$ represent the attenuation factor of the loss. If $\tilde{I}_{ij} > \tilde{I}_{tj}$ *or* $\tilde{I}_{ij} - \tilde{I}_{tj} > 0$ then we says it is the dominance degree of gain and $d(\tilde{I}_{ij},\tilde{I}_{tj})$ represent the distance of q-ROFF and $\tilde{w}_{jr}$ is relative weight of each criterion $\tilde{C}_j$. If $\tilde{I}_{ij} < \tilde{I}_{tj}$ *or* $\tilde{I}_{ij} - \tilde{I}_{tj} < 0$ then it represents the dominance degree of loss. If the interval is gain we use $\sqrt{\tilde{w}_{jr}\, d(\tilde{I}_{ij},\tilde{I}_{tj}) / \sum_{i=1}^{n}\tilde{w}_{jr}}$ but if the interval is loss we use $-\frac{1}{\tilde{\theta}}\sqrt{\sum_{i=1}^{n}\tilde{w}_{jr} d(\tilde{I}_{ij},\tilde{I}_{tj}) / \tilde{w}_{jr}}$. When $\tilde{I}_{ij} = \tilde{I}_{tj}$ it is nil. *n* is number of criteria; *j* be any criteria for $j = 1,2,...,n$;

$\tilde{w}_{rj}$ is equal to $\tilde{w}_j$ divided by $\tilde{w}_r$, where r is reference criteria.

$\tilde{I}_{ij}$ and $\tilde{I}_{tj}$ are respectively the performance of alternatives $\tilde{A}_i$ and $\tilde{A}_t$ in relation to *j*;

Step 5: Calculate the overall dominance degree of $\tilde{A}_i$ over each alternative $\tilde{A}_t$ using

$$\tilde{\delta}\left(\tilde{A}_i,\tilde{A}_t\right) = \sum_{j=1}^{n}\tilde{\phi}_j\left(\tilde{A}_i,\tilde{A}_t\right) \quad \forall\ (i,t)$$

$\tilde{\delta}\left(\tilde{A}_i,\tilde{A}_t\right)$ denotes the measurement of dominance of alternative $\tilde{A}_i$ over alternative $\tilde{A}_t$

Step 6: Derive the overall value of each alternative $\tilde{A}_i$ by using

$$\tilde{\psi}_i = \frac{\sum_{i=1}^{n}\tilde{\delta}\left(\tilde{A}_i,\tilde{A}_t\right) - \min_i\left(\sum_{i=1}^{m}\tilde{\delta}\left(\tilde{A}_i,\tilde{A}_t\right)\right)}{\max_i\left\{\sum_{i=1}^{m}\tilde{\delta}(\tilde{A}_i,\tilde{A}_t)\right\} - \min_i\left(\sum_{i=1}^{m}\tilde{\delta}\left(\tilde{A}_i,\tilde{A}_t\right)\right)}$$

Clearly, $0 \le \tilde{\psi}_i \le 1$, and we select the greater value of $\tilde{\psi}_i$ that is considered as a better alternative $\tilde{A}_i$. Thus, one can choose the appropriate alternative, in accordance with a descending order of the overall values of all the alternatives.

Step 7: Determine the ranking of the alternatives according to the overall values.

Now we apply our proposed method to deal with some daily life problems involving multicriteria decision making.

Example 3. **Selection of best ISP** (**Internet Service Provider**)

ISPs are essential in enabling user access to the internet and provide the infrastructure required for data transfer. To allow data to move between various devices and across the internet, they maintain networks of routers, switches, and other hardware. Comcast, AT&T, Verizon, Spectrum, Cox Communications, and many others are a few well-known ISPs. Our aim is to select the best ISP among the given ones. For this purpose, consider a customer who wants to choose an ISP for residential or commercial purposes. Suppose there are four types of an ISP (alternatives) $\tilde{A}_j\left(j = 1,\ 2,\ 3,\ 4\right)$ available. We need to look for the reliable offers at the most affordable price which would support long-term requirement. The customer considers four attributes to decide which ISP to choose.

$\tilde{C}_1$=Speed $\tilde{C}_2$= Consistency $\tilde{C}_3$ =Reliability $\tilde{C}_4$= Cost

The brief descriptions of the above four criteria $\tilde{C}_1,\tilde{C}_1,\tilde{C}_1$ and $\tilde{C}_1$ s are as follows:

$\tilde{C}_1$= Speed: The internet speed at which the data travels from the worldwide web to home computer, tablet and smart phone.

$\tilde{C}_2$= Consistency: Steadiness or uniformity in an internet service.

$\tilde{C}_3$ = Reliability: The internet service is consistently good in quality and performance.

$\tilde{C}_4$= Cost: The payment of the installations and the devices of internet service.

We notice that $\tilde{C}_4$ is cost attribute while the other three are benefit attributes. The values given by DM are displayed in Table 3.

Since $\tilde{C}_4$ is cost attribute, we have to convert it into benefit type by taking the complement as follows:

$$\tilde{C}_4^c = \left\{(0.8,0.7),(0.5,0.8),(0.7,0.6),(0.8,0.3)\right\}$$

Table 4 shows the normalized q-ROF decision matrix. Assume that the weights of criteria $\tilde{C}_j$, $j = 1,2,3,4$ are known and the corresponding weight vector is $\tilde{w} = \left(0.4,0.2,0.1,0.3\right)$. Since $\tilde{w}_1$ is maximum of all given weights so $\tilde{C}_1$ is considered as reference criterion and the corresponding reference weight is denoted by $\tilde{w}_1$=0.4. Therefore, the relative weights of all the criteria $\tilde{C}_j\left(j = 1,2,3,4\right)$ are as follow:

$$\tilde{w}_{1r}=\frac{0.4}{0.4}=1,\ \left(\tilde{w}_{jr}=\frac{\tilde{w}_j}{\tilde{w}_r}\right),\ \tilde{w}_{2r}=\frac{0.2}{0.4}=0.5,\ \tilde{w}_{3r}=\frac{0.1}{0.4}=0.25,\quad \tilde{w}_{4r}=\frac{0.3}{0.4}=0.75,\ \tilde{\theta}=\sum_{j=1}^{4}w_{jr}=2.5$$

Utilizing step 4, we calculate the dominance degree of the alternative $\tilde{A}_i$ over each alternative $\tilde{A}_t$ with respect to the criteria $\tilde{C}_j$. Since $\tilde{\theta}$ =2.5 for dominance degree, first we have to calculate the distance $\left(\tilde{I}_{ij},\tilde{I}_{tj}\right)$ for each criterion, so we use the similarity given in Equation (11)

$$\tilde{S}_{bacC}(\tilde{E},\tilde{F})=1-\frac{1}{3n}\sum_{i=1}^{n}\left|\mu_{\tilde{E}}^{q}(x_i)-\mu_{\tilde{F}}^{q}(x_i)\right|+\left|\nu_{\tilde{E}}^{q}(x_i)-\nu_{\tilde{F}}^{q}(x_i)\right|+\frac{1}{2}\left|\mu_{\tilde{E}}^{q}(x_i)-\mu_{\tilde{F}}^{q}(x_i)+\nu_{\tilde{F}}^{q}(x_i)-\nu_{\tilde{E}}^{q}(x_i)\right|$$

Table 5 shows the matrix for criterion $\tilde{C}_1$. Table 5 reflects the evaluation of the dominance degree of the alternatives $\tilde{A}_i$ over each alternatives $\tilde{A}_t$ with respect to criterion $\tilde{C}_1$ .Table 6 shows the matrix for criterion $\tilde{C}_2$. Table 6 shows the evaluation of the dominance degree of the alternatives $\tilde{A}_i$ over each alternative $\tilde{A}_t$ with respect to criterion $\tilde{C}_2$. Table 7 shows the matrix for criterion $\tilde{C}_3$. Table 7 exhibits the evaluation of the dominance degree of the alternatives $\tilde{A}_i$ over each alternative $\tilde{A}_t$ with respect to criterion $\tilde{C}_3$. Table 8 shows the matrix for criterion $\tilde{C}_4$. Table 8 denotes the evaluation of the dominance degree of the alternatives $\tilde{A}_i$ over each alternative $\tilde{A}_t$ with respect to criterion $\tilde{C}_4$.

The overall dominance degree of $\tilde{A}_i$ over each alternative $\tilde{A}_t$ is determined using $\tilde{\delta}\left(\tilde{A}_i,\tilde{A}_t\right)=\sum_{j=1}^{n}\phi_j\left(\tilde{A}_i,\tilde{A}_t\right)$. Table 9 shows the evaluation of the dominance degree of the alternatives $\tilde{A}_i$ over each alternative $\tilde{A}_t$ with respect to criterion $\tilde{C}_j$. The overall dominance degree of $\tilde{A}_i$ over each alternative $\tilde{A}_t$ is determined using $\tilde{\delta}\left(\tilde{A}_i,\tilde{A}_t\right)=\sum_{j=1}^{n}\phi_j\left(\tilde{A}_i,\tilde{A}_t\right)$.

Table 3. Pythagorean fuzzy decision-making matrix

	$\tilde{C}_1$	$\tilde{C}_2$	$\tilde{C}_3$	$\tilde{C}_4$
$\tilde{A}_1$	(0.8, 0.7)	(0.6, 0.9)	(0.9, 0.5)	(0.7, 0.8)
$\tilde{A}_2$	(0.8, 0.6)	(0.6, 0.8)	(0.9, 0.4)	(0.8, 0.5)
$\tilde{A}_3$	(0.6, 0.5)	(0.7, 0.4)	(0.5, 0.4)	(0.6, 0.7)
$\tilde{A}_4$	(0.5, 0.8)	(0.3, 0.7)	(0.5, 0.7)	(0.3, 0.8)

Table 4. The normalized q-ROF decision matrix

	$\tilde{C}_1$	$\tilde{C}_2$	$\tilde{C}_3$	$\tilde{C}_4$
$\tilde{A}_1$	(0.8, 0.7)	(0.6, 0.9)	(0.9, 0.5)	(0.8, 0.7)
$\tilde{A}_2$	(0.8, 0.6)	(0.6, 0.8)	(0.9, 0.4)	(0.5, 0.8)
$\tilde{A}_3$	(0.6, 0.5)	(0.7, 0.4)	(0.5, 0.4)	(0.7, 0.6)
$\tilde{A}_4$	(0.5, 0.8)	(0.3, 0.7)	(0.5, 0.7)	(0.8, 0.3)

Table 5. The matrix for criterion $\tilde{C}_1$

	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$	$\tilde{A}_4$
$\tilde{A}_1$	0.0000	-0.6132	0.0000	0.5374
$\tilde{A}_2$	0.6132	0.0000	0.5692	0.5138
$\tilde{A}_3$	0.0000	-0.5692	0.0000	0.5517
$\tilde{A}_4$	-0.5374	-0.5138	-0.5517	0.0000

Table 6. The matrix for criterion $\tilde{C}_2$

	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$	$\tilde{A}_4$
$\tilde{A}_1$	0.0000	-0.8443	-0.6951	0.3768
$\tilde{A}_2$	0.4221	0.0000	-0.7537	0.4052
$\tilde{A}_3$	0.3476	0.3768	0.0000	0.3742
$\tilde{A}_4$	-0.7537	-0.8104	-0.7483	0.0000

Table 7. The matrix for criterion $\tilde{C}_3$

	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$	$\tilde{A}_4$
$\tilde{A}_1$	0.0000	-1.2458	0.2588	0.2427
$\tilde{A}_2$	0.3114	0.0000	0.2642	0.2362
$\tilde{A}_3$	-1,0354	-1.0568	0.0000	0.2933
$\tilde{A}_4$	-0.9708	-0.9449	-1.1730	0.0000

Table 8. The matrix for criterion $\tilde{C}_4$

	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$	$\tilde{A}_4$
$\tilde{A}_1$	0.0000	0.4648	0.5049	-0.6701
$\tilde{A}_2$	-0.6198	0.0000	-0.6282	-0.5465
$\tilde{A}_3$	-0.6733	0.4712	0.0000	-0.6613
$\tilde{A}_4$	0.5026	0.4099	0.4959	0.0000

Table 9. Overall dominance degree of $\tilde{A}_i$ over each alternative $\tilde{A}_t$

	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$	$\tilde{A}_4$	$\sum_{i=1}^{4}\tilde{\delta}\left(\tilde{A}_i,\tilde{A}_t\right)$
$\tilde{A}_1$	0.0000	-1.2458	0.2588	0.2427	-1.6831
$\tilde{A}_2$	0.3114	0.0000	0.2642	0.2362	0.7871
$\tilde{A}_3$	-1,0354	-1.0568	0.0000	0.2933	-1.6385
$\tilde{A}_4$	-0.9708	-0.9449	-1.1730	0.0000	-5.5956

Table 10. Ranking the alternatives

	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$	$\tilde{A}_4$
Ψ_1	0.6129	1	0.6199	0

$$\Psi_i = \frac{\sum_{i=1}^{m}\tilde{\delta}\left(\tilde{A}_i,\tilde{A}_t\right) - \min_i \tilde{\delta}\left(\tilde{A}_i,\tilde{A}_t\right)}{\max_i\left\{\sum_{i=1}^{m}\tilde{\delta}\left(\tilde{A}_i,\tilde{A}_t\right)\right\} - \min_i \sum_{i=1}^{m}\tilde{\delta}\left(\tilde{A}_i,\tilde{A}_t\right)}.$$

Table 10 shows the ranking of alternatives. From Table 10 we conclude that the best alternative is $\tilde{A}_2$. The internet service provider $\tilde{A}_2$ is the best among all considered alternatives.

6. Conclusion and Future study

In this paper, we proposed a novel similarity measure and its axiomatic definitions of q-rung orthopair fuzzy sets based on below, above and center fuzzy sets. We have converted q-ROFSs into their below, above and center fuzzy sets and suggested a new method of calculating similarity measures between q-ROFSs. Based on numerically computed results, we have found that our proposed similarity measures between q-ROFS are suitable and logically reasonable. In the end, O-TODIM method is proposed to rank internet service providers and selects the best one/ones. Final results show the effectiveness and reliability of our proposed O-TODIM to solve a complex multicriteria decision making problem.

In the future, the proposed measures can be extended to Bipolar fuzzy sets, HFSs, and Picture fuzzy sets etc. The proposed measures can be applied to some more real-life problems.

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